Model Question Paper-1 with effect from 2019-20 (CBCS Scheme)

USN

Fourth Semester B.E. Degree Examination
Engineering Statistics & Linear Algebra

TIME: 03 Hours **Max. Marks: 100**

Note: 01. Answer any **FIVE** full questions, choosing at least **ONE** question from each **MODULE.**
02. Use of Normalized Gaussian Random Variables table is permitted.

		Module -1	***Bloom's Taxonomy Level**	**Marks**
Q.01	a	The PDF for the random variable Z is $$f_Z(z)=\begin{cases}\frac{1}{6\sqrt{z}} & ; \ 0<z<9\\ 0 & ; \ Otherwise\end{cases}$$ What are (i) the mean (ii) the mean of the square, and (iii) the variance of the random variable Z?	L1, L2	5
	b	Given the data in the following table, <table><tr><td>k</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>x_k</td><td>2.1</td><td>3.2</td><td>4.8</td><td>5.4</td><td>6.9</td></tr><tr><td>$P(x_k)$</td><td>0.21</td><td>0.18</td><td>0.20</td><td>0.22</td><td>0.19</td></tr></table>(i) Plot the PDF and the CDF of the discrete random variable X. (ii) Write expressions for PDF and CDF using unit-delta functions and unit – step functions.	L3	5
	c	Define an exponential random variable. Obtain the characteristic function of an exponential random variable and using the characteristic function derive its mean and variance.	L1, L3	10
		OR		
Q.02	a	It is given that $E[X]=36.5$ and that $E[X^2]=1432.3$ (i) Find the standard deviation of X. (ii) If $Y=4X-500$, find the mean and variance of Y.	L3	4
	b	Define a Poisson random variable. Obtain the characteristic function of a Poisson random variable and hence find mean and variance using the characteristic function.	L1, L3	10
	c	The random variable X is uniformly distributed between 0 and 4. The random variable Y is obtained from X using $y=(x-2)^2$.What are the CDF and PDF for Y?	L2, L3	6
		Module-2		
Q. 03	a	The joint PDF $f_{XY}(x,y)=c$, a constant, when $(0<x<3)$ and $(0<y<4)$ and is 0 otherwise. (i) What is the value of the constant c? (ii) What are the PDFs for X and Y? (iii) What is $F_{XY}(x,y)$when $(0<x<3)$ and $(0<y<4)$? (iv) What are $F_{XY}(x,\infty)$ and $F_{XY}(\infty,y)$? (v) Are X and Y independent?	L1, L2, L3	10
	b	Define correlation coefficient of random variables X and Y. Show that it is bounded by limits ± 1.	L1, L2	5

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Scheme and Solution Prepared By Prof. Basavraj Goudar

	c	X is a random variable with $\mu_X = 4$ and $\sigma_X = 5$. Y is a random variable with $\mu_Y = 6$, and $\sigma_Y = 7$. The correlation coefficient is 0.7. If U = 3X + 2Y, what are Var[U], Cov [UX] and Cov [UY]?	L3	5
		OR		
Q.04	a	X is a random variable uniformly distributed between 0 and 3. Y is a random variable, independent of X, uniformly distributed between +2 and −2. W = X + Y. What is the PDF for W?	L2, L3	8
	b	The random variable Z is uniformly distributed between 0 and 1. The random variable Y is obtained from Z as follows: Y = 3.5Z + 5.25 One hundred independent realizations of Y are averaged: $$V = \frac{1}{100}\sum_{i=1}^{100} Y_i$$ (i) Estimate the probability $P(V \leq 7.1)$ (ii) If 1000 independent calculations of V are performed, approximately how many of these calculated values for V would be less than7.1?	L3, L4	8
	c	Explain briefly the following random variables. (i) Chi-Square Random Variable (ii) Student's t Random Variable	L1	4
		Module-3		
Q. 05	a	With the help of an example, define Random Process and discuss distributions and density functions of a random process.	L1	5
	b	A random process is described by $$X(t) = A\cos(\omega_c t + \varphi + \theta)$$ Where A, ω_c and φ are constants and where θ is a random variable uniformly distributed between $\pm\pi$. Is $X(t)$ wide-sense stationary? If not, then why not? If so, then what are the mean and the autocorrelation function for the random process?	L2, L3	8
	c	Define the autocorrelation function (ACF) of a random process and discuss its properties.	L1, L2	7
		OR		
Q. 06	a	$X(t)$ and $Y(t)$ are independent, jointly wide-sense stationary random processes given by, $X(t) = A\cos(\omega_1 t + \theta_1)$ and $Y(t) = B\cos(\omega_2 t + \theta_2)$. If $W(t) = X(t)Y(t)$ then find the ACF $R_W(\tau)$.	L3	6
	b	Assume that the data in the following table are obtained from a windowed sample function obtained from an ergodic random process. Estimate the ACF for $\tau = 0, 2\ ms$ and $4\ ms$, where $\Delta t = 2\ ms$. $x(t)$: 1.5, 2.1, 1.0, 2.2, −1.6, −2.0, −2.5, 2.5, 1.6, −1.8 k: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9	L2, L3	6
	c	Suppose that the PSD input to a linear system is $S_X(\omega) = K$. The cross-correlation of the input X(t) with the output Y(t) of the linear system is found to be $$R_{XY}(\tau) = K\begin{cases} e^{-\tau} + 3e^{-2\tau}; & \tau \geq 0 \\ 0; & \tau < 0 \end{cases}$$ What is the power filter function $\lvert H(j\omega)\rvert^2$?	L3, L4	8
		Module-4		
Q. 07	a	Describe the column space and the null space of the following matrices. (i) $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$	L2, L3	4
	b	Determine whether the vectors (1, 3, 2), (2, 1, 3) and (3, 2, 1) are linearly dependent or independent.	L3	6

	c	If $u = \begin{bmatrix}1\\2\\2\end{bmatrix}$, $v = \begin{bmatrix}2\\-2\\1\end{bmatrix}$ and $w = \begin{bmatrix}2\\1\\-2\end{bmatrix}$ then show that u, v, w are pairwise orthogonal vectors. Find lengths of u, v, w and find orthonormal vectors u_1, v_1, w_1 from vectors u, v, w.	L2, L3	10
		OR		
Q. 08	a	Apply Gram-Schmidt process to $a = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $b = \begin{bmatrix}0\\1\\1\end{bmatrix}$ and $c = \begin{bmatrix}1\\1\\1\end{bmatrix}$ and write the result in the form of A = QR.	L3	8
	b	Find the dimension and basis for four fundamental subspaces for $$A = \begin{bmatrix}1 & 2 & 0 & 1\\0 & 1 & 1 & 0\\1 & 2 & 0 & 1\end{bmatrix}$$	L3	8
	c	Find the projection of b onto the column space of A. $A = \begin{bmatrix}1 & 1\\1 & -1\\-2 & 4\end{bmatrix}$ and $b = \begin{bmatrix}1\\2\\7\end{bmatrix}$	L3	4
		Module-5		
Q. 09	a	(i) Reduce the matrix A to U and find *det (A)* using pivots of A. $$A = \begin{bmatrix}1 & 2 & 3\\2 & 2 & 3\\3 & 3 & 3\end{bmatrix}$$ (ii) By applying row operations to produce an upper triangular matrix U, compute the *det (A)*. $$A = \begin{bmatrix}1 & 2 & 3 & 0\\2 & 6 & 6 & 1\\-1 & 0 & 0 & 3\\0 & 2 & 0 & 7\end{bmatrix}$$	L3	6
	b	Find the eigen values and eigen vectors of matrix A. $$A = \begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$$	L3	6
	c	Factor the matrix A into $A = X\Lambda X^{-1}$ using diagonalization and hence find A^3. $$A = \begin{bmatrix}1 & 2\\0 & 3\end{bmatrix}$$	L3	8
		OR		
Q. 10	a	Factorize the matrix A into $A = U\Sigma V^T$ using SVD. $$A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}$$	L3, L4	8
	b	(i) What is a positive definite matrix? Mention the methods of testing positive definiteness. (ii) Check the following matrix for positive definiteness. $$S_1 = \begin{pmatrix}5 & 6\\6 & 7\end{pmatrix}$$	L1, L2	6
	c	Find an orthogonal matrix Q that diagonalizes the following symmetric matrix. $$S = \begin{bmatrix}1 & 0 & 2\\0 & -1 & -2\\2 & -2 & 0\end{bmatrix}$$	L3	6

MODEL Question Paper

Subject: ESLA (18EC44). Sem: 04.
Name of the Faculty: Basavaraj D. Goudar.

Note: 1. Answer any 5 Full questions, choosing at least ONE question from each MODULE.
2. Use of Normalized Gaussian-RV's table is permitted.

Q1 a. The pdf of the RV-Z is given as

$$f_Z(z) = \begin{cases} \frac{1}{6\sqrt{z}} & 0 < z < 9 \\ 0 : & \text{otherwise.} \end{cases}$$

what are the i> mean ii> mean-square and iii> Variances of the RV-Z.? (5 Marks)

Ans: i> mean $\mu_Z = \int_0^1 z \frac{1}{6\sqrt{z}} dz = \frac{1}{6} \int_0^1 z^{1/2} dz$

$$= \frac{1}{6}\left[\frac{z^{3/2}}{3/2}\right]_0^9 = \frac{1}{6}\left[\frac{9^{3/2}}{3/2} - 0\right] = \frac{2}{3\times 6}\left[9^{3/2}\right]$$

$$= \frac{\cancel{2}^{1}}{\cancel{18}_{9}}\left[3^{\cancel{2}\times 3/\cancel{2}}\right] = \frac{1}{9}\left[3^3\right] = \frac{\cancel{27}^{3}}{\cancel{9}} = \underline{3}$$

ii> $E[Z^2] = \overline{Z^2} = \int_0^1 z^2 \frac{1}{6\sqrt{z}} dz$

$$= \frac{1}{6}\int_0^9 \left[z^{3/2}\right] dz$$

$$= \frac{1}{6}\left[\frac{z^{5/2}}{5/2}\right]_0^9 = \frac{1}{6}\left[\frac{3^{\cancel{2}\times 5/\cancel{2}}}{5/2}\right]$$

$$= \frac{\cancel{2}^{1}}{3\cancel{6}\times 5}\left[3^5\right] = \frac{1}{15} \times 243$$

$$\therefore \boxed{\overline{Z^2} = E[z^2] = 16.2}$$

iii> $\sigma_Z^2 = \overline{Z^2} - \mu_Z^2 = 16.2 - (3)^2 = 16.2 - 9$

$$\therefore \boxed{\sigma_Z^2 = 7.2}$$

(P.T.O)

Rough-work

$3\times3\times3\times3\times3$

27×9

81×3

243

Q 16. Given the following data in the table 16.
i) Plot pdf and cdf of the discrete RV-X. (10 marks)

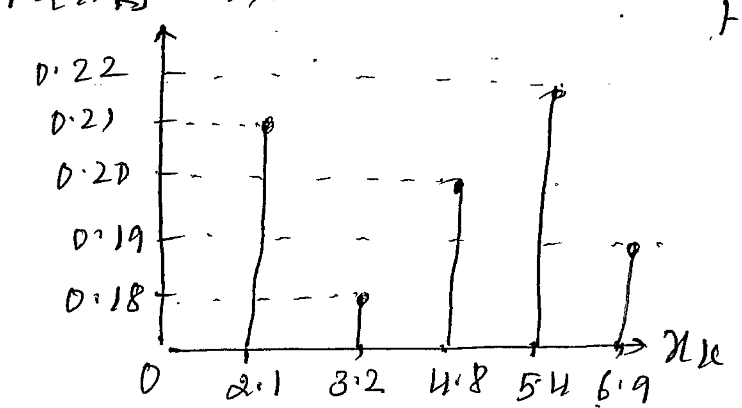
Ans:

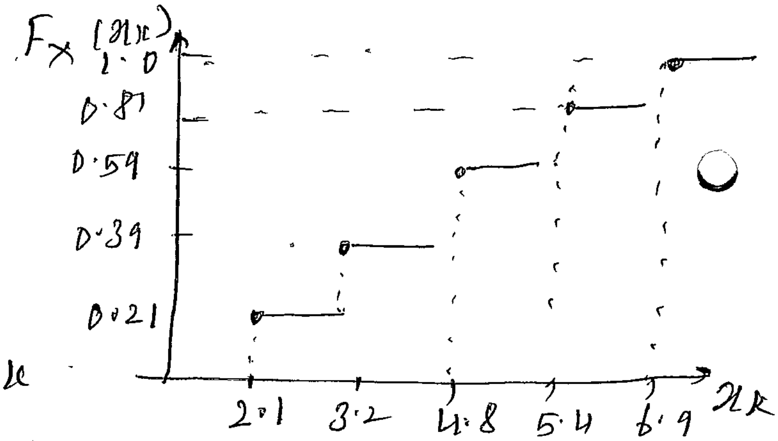
$k = i$	1	2	3	4	5
x_k	2.1	3.2	4.8	5.4	6.9
$P(X_k)$	0.21	0.18	0.20	0.22	0.19

ii) Write the expression for pdf and cdf using unit delta-functions & unit-step functions.

Ans: i) The plot of pdf and cdf of the discrete RV-X:

$k = i$	1	2	3	4	5
x_k	2.1	3.2	4.8	5.4	6.9
$P\{X_k\}$	0.21 (a)	~~.18~~ 0.18 (b)	0.20 (c)	0.22 (d)	0.19 (e)
$F_X(x_k)$	0.21	0.39	0.59	0.81	1.00
	↑ a	↑ a+b	↑ a+b+c	↑ a+b+c	↑ a+b+c+d





ii) Expressions for pdf and cdf using unit delta functions & unit-step functions

$$f_X(x_k) = 0.21\,\delta(x-2.1) + 0.18\,\delta(x-3.2) + 0.20\,\delta(x-4.8) + 0.22\,\delta(x-5.4) + 0.19\,\delta(x-6.9)$$

$$F_X(x) = 0.21\,u(x-2.1) + 0.18\,U(x-3.2) + 0.20\,u(x-4.8) + 0.22\,U(x-5.4) + 0.19\,U(x-6.9$$ Ans.

Q 1c. Define an exponential random variable. Obtain the characteristic function of an exponential-RV and using the characteristic function derive its mean & variance. (04 marks)

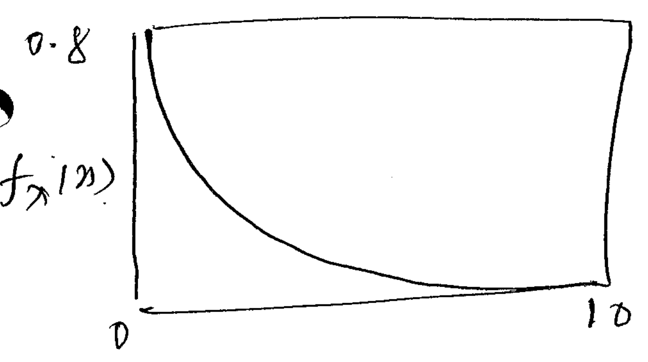
Ans: The exponential-distributed events of an expt is modeled as exponential-RV, and is widely used in the field of reliability.

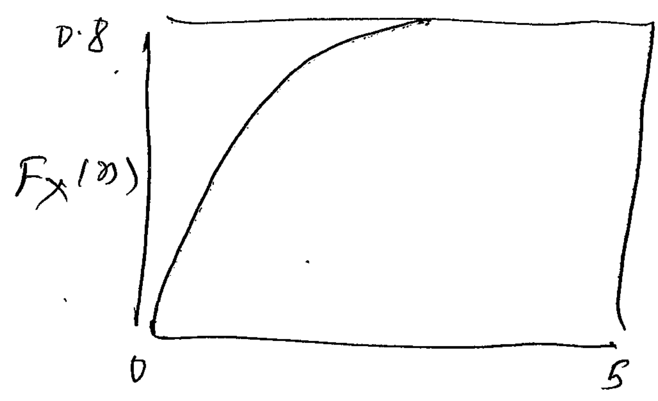
Consider a RV-X which is exponentially distributed with parameter λ. (failure-rate or the process rate), then it's pdf and cdf's are

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and $$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(referring to exponential examples in module I) plots of pdf and cdf is as shown below.





characteristic function of exponential-distribution

$$\phi_X(j\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} \lambda e^{-\lambda x} e^{j\omega x}\, dx$$

$$= \lambda \int_{0}^{\infty} e^{-(\lambda - j\omega)x}\, dx$$

(PTO)

$$= \lambda \left[\frac{e^{-(\lambda - j\omega)x}}{\lambda - j\omega} \right]_0^\infty$$

$$\Phi_X(j\omega) = \frac{\lambda}{\lambda - j\omega} \longrightarrow \text{Ans.}$$

Variance using the characteristic function
<u>mean-square</u>

$$\frac{d^2}{dj\omega}\left[\frac{\lambda}{\lambda - j\omega}\right]^2 = \lambda \frac{d}{dj\omega}(\lambda - j\omega)^{-2}$$

$$= \lambda(\lambda - j\omega)^{-3}(-1)$$

$$= \frac{2\lambda}{(\lambda - j\omega)^3}\Big|_{\omega=0}$$

$$= \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

<u>Variance</u>, using the formula that relate mean mean-square & variance

and i.e. $\sigma_x^2 = E[z^2] - \mu_2^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$ Ans.

Q 2 a. Given that $E[X] = 36.5$, $E[X^2] = 1432.3$ Find i) SD of X ii) If $Y = 4X - 500$ find the mean & variance of Y. (04 marks)

Ans: i) SD of X: W.k.t $\sigma_x^2 = E[X^2] - \mu_x^2$

$$= 1432.3 - (36.5)^2 = 100.05$$

$$\therefore \sigma_x = 10.0025 \rightarrow$$

$$\mu_Y = 4E[X] - 500$$

$$= 4 \times 36.5 - 500$$

$$\mu_Y = -354$$

$\left\{ \because Y = 4X - 500 \quad \therefore E[Y] = 4E[X] - 500 \right\}$

and variance $\sigma_Y^2 = a^2 \sigma_x^2 = 4^2 \times 100.05$

$$= 1600.8.$$

Q 2 b. Define a Poisson random variable. Obtain the characteristic function of a Poisson RV and hence find mean & variance using the characteristic function. (10 marks)

Soln: If X is a RV which follows a Poisson distribution with mean λ, then the pmf is given by the formula

$$P_X(k) = \frac{e^{-\lambda} \lambda^k}{k!} \quad k = 0, 1, 2, \cdots$$

and the cdf of X is

$$F_X(x) = e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!} \quad n \le x \le n+1.$$

The mean of the poisson distribution is $\mu = E[X] = \lambda$

The variance of the Poisson distribution is

$$\sigma = Var[x] = \lambda.$$

characteristic function of the Poisson-distribution is

$$\Phi_x(j\omega) = \exp[b(e^{j\omega}-1)]$$

The mean μ using characteristic function is

$$\frac{d}{d(j\omega)}(\Phi_x(j\omega)) = \frac{d}{d(j\omega)} e^{b(e^{j\omega}-1)}$$

$x = j\omega$

$$\frac{d}{dx}(\Phi_x(j\omega)) = e^{b(e^{x}-1)} \times b e^{x}$$

$$= e^{b(e^{j\omega}-1)} \times b e^{j\omega} \Big|_{x=j\omega}$$

$$= e^{b(e^{0}-1)} b \cdot e^{0}$$

$$= e^{b(1-1)} b \times 1$$

$$= e^{0} \times b \times 1 = b\mu \text{ Ans}$$

variance σ^2, using characteristic function

$$\frac{d^2}{dx^2}(x) = e^{b(e^{x}-1)} b e^{(x)}$$

$$= b e^{x}[e^{b(e^{x}-1)}] + b e^{x} e^{b.(e^{x}-1)} b e^{x}$$

$x = j\omega$.

$$= b e^{j\omega}[e^{b(e^{j\omega}-1)}] + b e^{j\omega} e^{b(e^{j\omega}-1)} b \times e^{j\omega}$$

$\omega = 0$

$$= b e^{0}[e^{b(e^{0}-1)}] + b e^{0} e^{b(0-1} e^{0} b$$

$$= b[e^{b(1-1)}] + b e^{b(1-1)} b(1)$$

$$= b + b \cdot b$$

$$= b + b^2$$

Q 2 c The RV - X is uniformly distributed betwn 0 and 4. The RV - Y is obtained from X using $y = (x-2)^2$. Find CDF & PDF for Y

(06 Mar

Soln:

$$f_X(x) = \begin{cases} \frac{1}{4} = 0.25 & 0 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = 0.25x \qquad 0 < x \leq 4$$

$$y = (x-2)^2$$

$$(x-2) = \pm\sqrt{y}$$

$$x = \pm\sqrt{y} + 2$$

$$F_Y(y) = P\{Y \leq y\}$$

$$= P\{-Y \leq y\}$$

$$x = \pm\sqrt{y} + 2$$

Q 3 a The joint-pdf $f_{XY}(x, y) = c$ a constant, when $0 < x < 3$ and $0 < y < 4$, and is 0 otherwise

i) what is the value of the constant c?

ii) what are the pdf for X and Y.

iii) what is $F_{XY}(x,y)$ when $0 < x < 3$ & $0 < y < 4$

iv) what are $F_{XY}(x, \infty)$ & $F_{XY}(\infty, y)$

v) Are X and Y are independent. (10 marks)

Soln: i) Given $f_{XY}(x,y) = c$, is the joint pdf

so $\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$

$$\int_0^4 \int_0^3 c\, dx\, dy = c \int_0^4 \left[\int_0^3 dx\right] dy$$

$$= c \int_0^4 [x]_0^3\, dy$$

$$1 = c \int_0^4 3\, dy = 3c\, [y]_0^4 = 3c \times 4 = 12c$$

$$\therefore \boxed{c = 1/12}$$

ii) $f_X(x) = c \int_0^4 1\, dy$ $\left\{ \text{using marginal pdf formula's} \right\}$

$= c\, [y]_0^4 = c \times 4$

$= \frac{1}{12} \times 4 = 1/3$ $(\because c = 1/12)$

$\therefore f_X(x) = 1/3$

similarly $f_Y(y) = c \int_0^3 1\, dx = c\, [3] = 3c$

$= \frac{1}{12} \times 3 = 1/4$. $0 < y < 4$.

iii) $F_{XY}(x,y) = c \int_0^x \int_0^y du\, dv$

(PTO)

3a continued ....

$$= c\int_0^x [u]_0^y \, dv = cy \int_0^x dv = cy\,[v]_0^x$$

$$= cyx = \frac{1}{12} \; //$$

iv) $F_X(x) = F_{XY}(x, \infty) = c\int_0^x \int_0^4 du\,dv$

$$= c\int_0^x \left[\int_0^4 du\right] dv$$

$$= c \int_0^x [y]_0^4 \, dv$$

$$= 4c\int_0^x dv = 4c\,[v]_0^x$$

$$= 4 \times \frac{1}{12} x = x/3 \qquad 0 < x < 3.$$

$F_Y(y) = F_{XY}(\infty, y) = c\int_0^3 \int_0^y du\,dv$

$$= c\int_0^3 \left[\int_0^y du\right] dv$$

$$= c \int_0^3 [y]_0^y \, dv$$

$$= yc\int_0^3 dv = yc\,[v]_0^3$$

$$= 3cy = 3\frac{1}{12}y = y/4 \qquad 0 < y < 3$$

v) From the above equations

$$f_X(x)\, f_Y(y) = f_{XY}(x,y)$$

$\therefore$ X and Y are independent.

Similarly it is observed that

$$F_X(x)\, F_Y(y) = F_{XY}(x,y) \; //.$$

Q 3b. Define Correlation-coefficient of RV's X and Y. Show that it is bounded by the limits ± 1. (05 Marks)

Ans: The Correlation coefficient denoted by ρ_{xy} is defined as

$$\rho_{xy} = \frac{Cov[xy]}{\sigma_x \sigma_y}$$

Proof for Limits: $\rightarrow$

$$E\left[\left(\frac{x-\mu_x}{\sigma_x} \pm \frac{y-\mu_y}{\sigma_y}\right)^2\right] \geq 0$$

$$= E\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \pm 2\,\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y}\right] \geq 0$$

$$= \frac{1}{\sigma_x} E[(x-\mu_x)^2] + \frac{1}{\sigma_y} E[(y-\mu_y)^2] \pm \frac{2}{\sigma_x \sigma_y} E[(x-\mu_x)(y-\mu_y)] \geq 0$$

$$= \frac{\sigma_x}{\sigma_x} + \frac{\sigma_y}{\sigma_y} \pm \frac{2}{\sigma_x \sigma_y} Cov[xy] \geq 0$$

$$= 1+1 \pm 2\rho_{xy} \geq 0$$

$$\boxed{|\rho_{xy}| \leq \pm 1}$$ proof.

Q 3C X is a RV having $\mu_x = 4$, $\sigma_x = 5$. Y is another RV having $\mu_y = 6$, $\sigma_y = 7$. The correlation coefficient is 0.2. If $U = 3X + 2Y$. what are the $Var[U]$, $Cov[UX]$ and $Cov[UY]$? (5 marks)

Soln: $Var[U] = \sigma_U^2 = E[(U - \mu_U)^2]$

$$= E[9(X - \mu_x)^2 + 12(X - \mu_x)(Y - \mu_y) + 4(Y - \mu_y)]$$

$$= 9\sigma_x^2 + 12\, Cov[XY] + 4\sigma_y^2$$

$$= 9 \times 5^2 + 12(7) + 4(7^2)$$

$$= 505$$

b) $Cov[UX] = E[(U - \mu_U)(X - \mu_x)]$

$$= E[\{3(X - \mu_x) + 2(Y - \mu_y)\}(X - \mu_x)]$$

$$= 3\sigma_x^2 + 2\, Cov[XY]$$

$$= 3(5^2) + 2(7) = 75 + 14 = 89.$$

c) $Cov[UY] = E[(U - \mu_U)(Y - \mu_y)]$

$$= E[\{3(X - \mu_x) + 2(Y - \mu_y)\}(Y - \mu_y)\}$$

$$= 3\, Cov[XY] + 2\sigma_y^2$$

$$= 3(7) + 2(7^2) = 21 + 98$$

$$= 119$$ Ans.

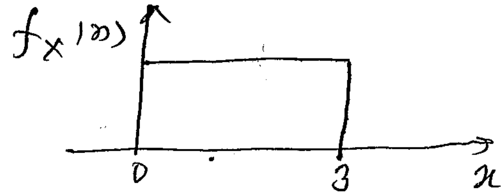
Q 4a RV-X is uniformly distributed between 0 and 3. Z is a RV independent of X, and uniformly distributed btw +1 and -1. $U = X + Z$. Find pdf of U.

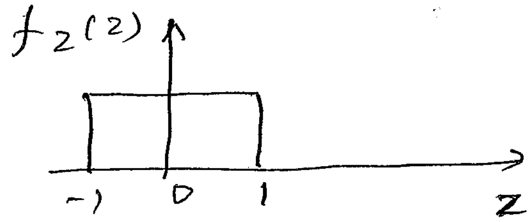
Soln:

$$f_X(x) = \begin{cases} \frac{1}{3} & 0 < x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{1}{2} & -1 < z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Case I: $-1 < u < 1$





$$f_U(u) = \int_{-\infty}^{\infty} f_Z(z)\, f_X(u-z)\, dz$$

$$= \int_{-1}^{u} \frac{1}{2} \times \frac{1}{3}\, dz$$

$$= \frac{1}{6}\left[z\right]_{-1}^{u} = \frac{1}{6}(u+1)$$

$$\therefore f_U(u) = \frac{u+1}{6} \qquad -1 < u < 1$$

Case II Width of the window $3 - 0 = 3$ i.e

$1 < u < 2$

$$f_U(u) = \int_{-\infty}^{\infty} f_Z(z)\, f_Z(u-z)\, dz$$

$$= \int_{-1}^{1} \frac{1}{2} \times \frac{1}{3}\, dz$$

$$= \frac{1}{6}\left[z\right]_{-1}^{1} = \frac{1}{6}\left[1-(-1)\right]$$

$$= \frac{1}{3} \qquad 1 < u < 2$$

(PTO)

Case 3: $2 < u < 4$.

$$f_U(u) = \int_{-\infty}^{\infty} f_Z(z)\, f_X(u-z)\, dz$$

$$= \int_{u-3}^{1} \frac{1}{2} \times \frac{1}{3}\, dz$$

$$= \frac{1}{6}\,[z]_{u-3}^{1} = \frac{1}{6}\,[1-(u-3)]$$

$$= \frac{4-u}{6} \qquad 2 < u < 4.$$

Q 4b. RV-Z is uniformly distributed betw 0 and 1. The RV-Y is obtained from Z as follows: $Y = 3Z + 5.5$

One hundred independent ~~real~~ realizations of Y are averaged

$$U = \frac{1}{100} \sum_{i=1}^{100} Y_i$$

i) Estimate the probability $P\{U \le 7.1\}$

ii) If 1000 independent calculations of U are performed, then approximately how many of these calculated values for U would be less than 7.1?

Soln: $\mu_Z = \dfrac{a+b}{2} = \dfrac{0+1}{2} = 0.5$

$$\sigma_Z^2 = \frac{b-a}{12} = \frac{1-0}{12} = 1/12$$

$$\mu_Y = 3\mu_Z + 5.5 = 3(0.5) + 5.5 = 7$$

$$\sigma_Y^2 = 3^2 \sigma_Z^2 = 9/12 = 3/4.$$

(PTO)

4b continued...

$$\mu_V = \mu_y = 7$$

$$\sigma_V = \sqrt{\frac{9}{1200}} = 0.0866.$$

ii> The probability $P\{V \le 7.1\}$

$$P\{V \le 7.1\} = F_V(7.1) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{7.1-7}{0.0866}\right)$$

$$= \Phi(1.1547) \quad \text{From Z-table}$$

$$\sigma_y^2 = 0.8759$$

iii> $P\{V \le 7.1\} \times 1000 = 876$ //

Q 4C. Explain briefly

i> Chi-square RV

ii> Student-t RV.

Ans: i> Chi-square RV.

The RV V where for integer for $r \geq 1$

$$V = \sum_{i=1}^{r} Z_i^2$$

w.k.t $\quad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

and : $E[Z] = \mu_Z = 0$

$$E[Z^2] = \sigma_Z^2 = 1$$

Now consider new RV $y = z^2$

(PTO)

conditional joint pdf for Y is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y^2/2} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$E[2^2] = \mu_Y = 1$

$E[Y^2] = E[2^4] = 3$

$\sigma_Y^2 = E[Y^2] - \mu_Y^2 = 2$ //.

$\phi\, j\omega = (1 - j2\omega)^{-1/2}$

when $r = 1$, $V_1 = Y_1 = Y$

$f_{V_1}(v) = f_Y(y)$

$E[V_1] = \mu_Y = 1$

$Var[V_1] = \sigma_Y^2 = 2$

$\phi_{V_1}(j\omega) = (1 - j2\omega)^{-1/2}$

when $r = 2$ $V_2 = 2_1 + 2_2 = Y_1 + Y_2 = V_1 + Y$

$E[V_2] = 2\mu_Y = 2$

$Var[V_2] = 2\sigma_Y^2 = 4.$

$\phi_{V_2}(j\omega) = \phi_{V_1}(j\omega)^2 = (1 - j2\omega)^{-1}$

when $r=3$, $2_1 + 2_2 + 2_3 = V_3 = V_2 + Y$

$E[V_3] = E[V_2] + \mu_Y = 3$

$Var[V_3] = Var[V_2] + \sigma_Y^2 = 6$

$\phi_{V_3}(j\omega) = [\phi_Y(j\omega)]^3 = (1 - j2\omega)^{-3/2}$

$r \geq 1$

$$\therefore f_V(v) = \begin{cases} \frac{1}{\Gamma(r/2)\, 2^{r/2}} v^{(\frac{r}{2}-1)} e^{-v/2} & v > 0 \\ 0 & v < 0 \end{cases}$$

4c continued - - - -

ii) Students t RV.

The RV - T where for $r \geq 1$

$$T = \frac{Z}{\sqrt{V/r}}$$

joint pdf

$$f_{TV} = \int_0^\infty f_{TV}(t, v)\, dv$$

by exchanging (t, v) with (x, y)

$$f_{TV} = \int_0^\infty f_T(t|v)\, f_V(v)\, dv$$

$$T = \sqrt{\frac{r}{V}}\, Z$$

$$f_T(t|v) = \sqrt{\frac{v}{r}}\, f_Z\left(\sqrt{v/r}\, t\right)$$

$$f_T(t|v) = \sqrt{\frac{v}{2\pi r}}\, e^{-(t^2/r)(v/2)}$$

$$\therefore f_T(t) = \frac{1}{\sqrt{2\pi r}\ \Gamma(r/2)\ (2^{r/2})} \int_0^\infty v^{(r+1)/2 - 1}$$

$$\times\ e^{-(1+t^2/r)(v/2)}\, dv$$

Let $u = (1 + t^2/r)(v/2)$

$$f_T(t) = \frac{1}{\sqrt{2\pi r}\ \Gamma(r/2)\ (1+t^2/r)^{(r+1)/2}}$$

$$\times \int_0^\infty w^{[(r+1)/2 - 1]} e^{-w}\, dw$$

$$\therefore f_T(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\ \Gamma(r/2)\ (1+t^2/r)^{(r+1)/2}}$$

Ans.

Q 5 a. With the help of an example, define RP & discuss distributions & density functions of a RP
(05 marks) (08M)

Ans: If we let a RV be a function of independent variable like time, space or any other types then that RV is referred to as RP.

eg: X is a RV

$X(t) \rightarrow RP$

We assume that when $t=0$, there exist a pdf,

$$f_{X(0)}(x) \geq 0 \quad -\infty < x < \infty \quad (1)$$

Similarly bivariate-pdf's exist so that for any pair of times t_1 and t_2 we have

$$f_{X(t_1)X(t_2)}(x_1, x_2) \geq 0 \quad -\infty < x_1, x_2 < \infty \rightarrow (2)$$

the bivariate case can also be applied to the two RP's $X(t)$ and $Y(t)$

$$f_{X(t_1)Y(t_2)}(x, y) \geq 0 \quad -\infty < x, y < \infty \rightarrow (3)$$

For multivariate of order n

$$f_{X(t_1)X(t_2)\cdots X(t_n)}(x_1, x_2, \cdots x_n) \geq 0 \quad -\infty < x_1, x_2, \cdots x_n < \infty \rightarrow (4)$$

For jointly two RP's $X(t)$ & $Y(t)$ multivariate pdf's exist ie

$$f_{X(t_1)X(t_2)\cdots X(t_n)Y(t_1')Y(t_2')\cdots Y(t_n')}(x_1, x_2, \cdots x_n, y_1, y_2, \cdots y_n) \geq 0 \longrightarrow (5)$$

Note: By using the standard relation betn cdf and pdf ie

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dy$$

we can obtain corresponding cdf's using the pdf's in eqn one to five.

Q 5b. A RP-X(t) is described by

$$X(t) = A \cos(\omega_c t + \varphi + \theta)$$

where A, ω_c and $\varphi \rightarrow$ are constants and $\theta \rightarrow$ is a RV. uniformly-distributed between $\pm \pi$.

Is X(t) is WSS? If not then why not? and if so, then what are the mean and ACF.

Soln: → The phase of the given RP-X(t) varies uniformly betn $\pm\pi$, thus we can define it's pdf as

$$f_\phi(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & \text{otherwise.} \end{cases}$$

There are two testing for a RP. to be as a WSS-R.P.

1st Test: Mean of the RP-X(t) must be independent of time t.

$$E[X(t)] = \mu_{X(t)} = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx$$

where $x = x(t) = A \cos(\omega_c t + \varphi + \phi)$

$$\therefore \mu_{X(t)} = \int_{-\pi}^{\pi} A \cos(\omega_c t + \varphi + \phi) \frac{1}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi} \int_{-\pi}^{\pi} \cos(\underbrace{\omega_c t + \varphi}_{A} + \underbrace{\phi}_{B})\, d\theta$$

using the formula $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\mu_{X(t)} = \frac{A}{2\pi} \int_{-\pi}^{\pi} [\cos(\omega_c t + \varphi) \cos\phi - \sin(\omega_c t + \varphi) \sin\phi]\, d\theta$$

$$= \frac{A}{2\pi} \cos(\omega_c t + \varphi) [\sin\phi]_{-\pi}^{\pi} - \frac{A}{2\pi} \sin(\omega_c t + \varphi) [\cos\phi]_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi} \cos[\omega_c t + \varphi] [\sin\pi - \sin(-\pi)] - \frac{A}{2\pi} \sin(\omega_c t + \varphi) [-1 - (-1)]$$

$$= \frac{A}{2\pi} \cos(\omega_c t + \varphi)(0 - 0) - \frac{A}{2\pi} \sin(\omega_c t + \varphi) [-1 + 1]$$

$$= \frac{A}{2\pi} \cos(\omega_c t + \varphi) \times 0 - \frac{A}{2\pi} \sin(\omega_c t + \varphi) \times 0$$

$\therefore \mu_{X(t)} = 0$ // $\therefore$ Test-I passed //.

Test II: ACF must be independent of the absolute time t and only depends on time-difference τ.

$$R_X(\tau) = E[X(t)\, X(t+\tau)]$$

$$= E[A\cos(\omega_c t+\psi+\phi)\, A\cos[\omega_c(t+\tau)+\psi+\phi]]$$

using the formula $\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$

$$= A^2 E[\cos(2\omega_c t+2\psi+2\phi+\omega_c\tau)+\cos(\omega_c\tau)]$$

$$\therefore R_X(\tau) = \frac{A^2}{4} E[\cos(2\omega_c t+2\psi+2\phi+\omega_c\tau)+\cos(\omega_c\tau)]$$

$$= \frac{A^2}{4}\int_{-\pi}^{\pi}[\cos(2\omega_c t+2\psi+2\phi+\omega_c\tau)+\cos(\omega_c\tau)]\frac{1}{2\pi}\,d\phi$$

$$= \frac{A^2}{4\pi}\Big\{[\sin(2\omega_c t+2\psi+2\phi+\omega_c\tau)]_{-\pi}^{\pi} + \cos\omega_c\tau\,[\pi-(-\pi)]$$

$$= \frac{A^2}{4\pi}[\sin(2\omega_c t+2\psi+2\phi+\omega_c\tau)]_{-\pi}^{\pi} + \frac{A^2}{4\pi}\cos\omega_c\tau \times 2\pi$$

$$= \frac{A^2}{4\pi}\times 0 + \frac{A^2}{2\cancel{4\pi}}\times\cancel{2\pi}\cos\omega_c\tau$$

$$\therefore R_X(\tau) = \frac{A^2}{2}\cos\omega_c\tau \longrightarrow \text{Ans.}$$

Referring to above equation ACF of the given RP- $X(t)$ is only the function of τ, not t

i.e. it passed in test-II also;

Hence the given RP - $X(t) = A\cos[\omega_c t+\psi+\phi]$ is a WSS RP with mean $= 0$

and ACF $R_X(\tau) = \frac{A^2}{2}\cos\omega_c\tau$.

Q 5 c. Define ACF of a RP, and discuss its ~~prope~~ properties. (07 Marks)

Ans: The ACF of a RP $X(t)$ is defined as

$$R_X(\tau) = E[X(t)\, X(t+\tau)]$$

where t and τ are arbitrary.

Note: ACF at $\tau=0$, $R_X(0) = E[X^2(t)]$ gives average square value of a R.P (PTO)

Properties of ACF

Property 1: The ACF evaluated at $\tau = 0$ is the average of the RP-squared. i.e $R_X(0)$ in a generalized sense called "an average power".

Moving another one step further, we know the relation betw mean, mean-square & variance

i.e
$$\sigma_{X(t)}^2 = E[X_{(t)}^2] - \mu_{X(t)}^2$$

or
$$\underset{\uparrow}{E[X_{(t)}^2]} = \underset{\uparrow}{\sigma_{X(t)}^2} + \underset{\uparrow}{\mu_{X(t)}^2}$$

Average power = Ac-power + average dc-power.

Property 2: Suppose that instead of τ, if we use $-\tau$ in the definition of ACF, then we have

$$R_X(-\tau) = E[X(t)\, X(t-\tau)]$$

Then since absolute time does not affect the ACF ($\because$ ACF is independent of t), we can introduce the change of variables $\tau' = t - \tau$ this leads to $\tau = t - \tau'$, $t = \tau' + \tau$)

$$R_X(-\tau) = E[X_{(\tau'+\tau)}\, X_{(\tau')}] \longrightarrow (B)$$

$\because$ $t = \tau + \tau'$ and

$t - \tau = \tau'$

But eqn (B) is same as

$$R_X(-\tau) = E[X(\tau')\, X(t'+\tau)]$$

$$\therefore \boxed{R_X(-\tau) = R_X(\tau)}$$

ie ACF is an even function of τ.

Property 3: $\rightarrow$ $|R_X(\tau)| \leq R_X(0)$

Property 4: $\rightarrow$ If $X(t)$ has periodic-component then $R_X(\tau)$ will also have periodic component of same period.

Property 5: $\rightarrow$ $R_X(\tau)$ has well defined shape

Q. 6 a. $X(t)$ and $Y(t)$ are independent, jointly WSS RP's given by $X(t) = A\cos(\omega_1 t + \phi_1)$
$Y(t) = B\cos(\omega_2 t + \phi_2)$ (OBM)
if $W(t) = X(t)\,Y(t)$, then find ACF $R_W(\tau)$.

Ans: → ACF of the RP $X(t)$ can be obtained as follows:

$$R_X(\tau) = E[X(t)\,X(t+\tau)]$$

$$= E[A\cos(\omega_1 t + \phi)\,A\cos[\omega_1 t + \omega_1\tau + \phi]]$$

$$= A^2 E[\cos(\omega_1 t + \phi)\cos(\omega_1 t + \omega_1\tau + \phi)]$$

using the formula $\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$= \frac{A^2}{2} E[\cos(\omega_1 t + \phi + \omega_1 t + \omega_1\tau + \phi) + \cos[\cancel{\omega_1 t} + \cancel{\phi} + \omega_1\tau - \cancel{\omega_1 t} - \cancel{\phi}]]$$

$$= \frac{A^2}{2} E[\cos(2\omega_1 t + 2\phi + \omega_1\tau) + \cos(\omega_1\tau)]$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi}[\cos(2\omega_1 t + 2\phi + \omega_1\tau) + \cos(\omega_1\tau)]\,d\theta$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi}\cos(2\omega_1 t + \omega_1\tau + 2\phi)\,d\theta + \frac{A^2}{2}\int_{-\pi}^{\pi}\cos(\omega_1\tau)\,d\theta$$

$$= 0 + \frac{A^2}{2}\cos(\omega_1\tau) \times [\pi - (-\pi)]$$

$$\therefore R_X(\tau) = \frac{A^2}{2}\cos\omega_1\tau$$

similarly $R_Y(\tau) = \frac{B^2}{2}\cos\omega_2\tau.$

when $W(t) = X(t)\,Y(t)$

then $R_W(\tau) = R_X(\tau)\,R_Y(\tau)$

$$= \frac{A^2}{2}\cos\omega_1\tau * \frac{B^2}{2}\cos\omega_2\tau$$

$$\therefore R_W(\tau) = \frac{A^2B^2}{4}\cos\omega_1\tau\cos\omega_2\tau$$

$$\Rightarrow R_W(\tau) = \frac{A^2B^2}{8}[\cos(\omega_1+\omega_2)\tau + \cos(\omega_1-\omega_2)\tau]$$ //

Ans.

Q 6 b. Assume that the data for the following table are obtained from a windowed-sample function obtained from an ergodic-RP. Estimate the ACF for $\tau = 0, 2$ ms & 4 ms. where $\Delta t = 2$ ms. (06M)

$x(t)$	1.5	2.1	1.0	2.2	-1.6	-2.0	-2.5	2.5	1.6	-1.8
k	0	1	2	3	4	5	6	7	8	9

Soln: The ACF for ergodic-process is given by

$$R_x(\tau) = \frac{1}{N\Delta t} \sum_{k=0}^{N-1-\tau} X_T(k\Delta t)\, x_T(k\Delta t+\tau) \cdot \Delta t \; ; \tau \geq 0$$

where $\Delta t \rightarrow$ sampling interval

Given $\tau = 0, 2, 4$ msec

$\Delta t = 2$ ms and $N = 10$

$$R_x(0) = \frac{1}{10 \times \cancel{2ms}} \sum_{k=0}^{10-1-0} X_T(k\Delta t)\, X_T(k\Delta t + \tau) \cdot \cancel{2ms}$$

$$= \frac{1}{10} \sum_{k=0}^{9} x_T(k\Delta t)\, x_T(k\Delta t+\tau)$$

$$R_x(0) = \frac{1}{10} \{ (1.5)(1.5) + (2.1)(2.1) + (1.0)(1.0) + (2.2)(2.2) + (-1.6)(-1.6) + (-2.0)(-2.0) + (-2.5)(-2.5) + (2.5)(2.5) + (1.6)(1.6) + (-1.8)(-1.8) \}$$

$$R_x(0) = 3.736$$

(PTO)

6 b continued...

similarly

$$R_x(2ms) = \frac{1}{N \cancel{\Delta t}} \sum_{k=0}^{10-1-\tau} x_T(k\Delta t)\, x_T(k\Delta t+\tau)\, \cancel{\Delta t}$$

Here take $\tau = 1$

$\because$ 2nd interaction, $N = 10$ and $\Delta t = 2ms$

$$R_x(2ms) = \frac{1}{10} \sum_{k=0}^{10-1-1} x_T(k\Delta t)\, x_T(k\Delta t+1)$$

$$= \frac{1}{10} \sum_{k=0}^{8} x_T(k\Delta t)\, x_T(k\Delta t+1)$$

$$= \frac{1}{10} [(1.5)(2.1) + (2.1)(1.0) + (1.0)(2.2) + (2.3)(-1.6) + (-1.6)(-2.0) + (-2.0)(-2.5) + (-2.5)(2.5) + (2.5)(1.6) + (1.6)(-1.8)]$$

$$R_x(2ms) = \frac{1}{10} \{ 3.15 + 2.1 + 2.42 - 3.68 + 3.2 + 5 - 6.25 + 4 - 2.88]$$

$$= \frac{1}{10} [7.06]$$

$$\therefore \boxed{R_x(2ms) = 0.706}$$

Similarly,

$$R_x(4ms) = \frac{1}{N \cancel{\Delta t}} \sum_{k=0}^{N-1-\tau} x_T(k\Delta t)\, x_T(k\Delta t+\tau)\, \cancel{\Delta t}$$

Here $\tau = 2$ ($\because$ 3rd iteration)

$$R_x(4ms) = \frac{1}{10} \sum_{k=0}^{10-1-2} x_T(k\Delta t)\, x_T(k\Delta t+\tau)$$

$$= \frac{1}{10} \sum_{k=0}^{7} x_T(k\Delta t)\, x_T(k\Delta t + \tau)$$

$$= \frac{1}{10} \Big[(1.5)(1.0) + (2.1)(2.2) + (1.0)(-1.6)$$
$$+ (2.2)(-2.0) + (-1.6)(-2.5) + (-2.0)(2.5)$$
$$+ (-2.5)(1.6) + 2.5)(-1.8) \Big]$$

$$R_x(4ms) = -0.9388 \,//$$

Extra ...
Problem 6b. text.

Q 6C. Suppose that the PSD-input to a linear system is $S_X(\omega) = K$. The cross-correlation of the input $X(t)$ with the output $Y(t)$ of the linear system is found to be

$$R_{XY}(\tau) = K \begin{cases} e^{-\tau} + 3e^{-2\tau} & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

what is the power filter function $|H(j\omega)|^2$?

Ans: → given $S_X(\omega) = K$

$$\therefore \quad R_X(\tau) = K\,\delta(\tau)$$

w.k.t $R_{XY}(\tau) = \int_0^{\infty} h(u)\, R_X(\tau - u)\, du$

$$= \int_0^{\infty} h(u)\, K\,\delta(\tau - u)\, du$$

$$R_{XY}(\tau) = K\,h(\tau)$$

$$h(t) = \begin{cases} 3e^{-\tau} + e^{-2\tau} & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

w.k.t $e^{-at}\,u(t) \longleftrightarrow \dfrac{1}{s+a}$ or $\dfrac{1}{j\omega + a}$

$$3e^{-t}\,u(t) \longleftrightarrow \frac{3}{s+1}$$

$$e^{-2t}\,u(t) \longleftrightarrow \frac{1}{s+2}$$

$\therefore$ Transfer function

$$H(s) = \left(\frac{3}{s+1}\right) + \left(\frac{1}{s+2}\right)$$

$$= \frac{3(s+2) + (s+1)\,1}{(s+1)\,(s+2)}$$

(PTO)

$$H(s) = \frac{4s+7}{s^2+3s+2}$$

$s = j\omega$

$$H(j\omega) = \frac{4(j\omega)+7}{(j\omega)^2+3(j\omega)+2}$$

$$= \frac{4j\omega+7}{-\omega^2+3j\omega+2}$$

$$H(-j\omega) = \frac{-4j\omega+7}{-\omega^2-3j\omega+2}$$

$$\therefore |H(j\omega)|^2 = H(j\omega)\, H(-j\omega)$$

$$= \left(\frac{4j\omega+7}{-\omega^2+3j\omega+2}\right)\left(\frac{-4j\omega+7}{-\omega^2-3j\omega+2}\right)$$

$$\boxed{|H(j\omega)|^2 = \frac{16\omega^2+49}{\omega^4+5\omega^2+4}}$$ Ans.

Q.7 a. Describe the column-space and the null space of the following

i) $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. ii) $B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

Q 7b: → Determine whether the vectors $(1,3,2)$, $(2,1,3)$ and $(3,2,1)$ are linearly dependent or independent.

Soln: -

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{5} R_2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$$

$S(A) = 3 =$ No of variables $= 3$.

$\therefore$ Given vectors are linearly - independent

Q7C. If $u = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$; then show that u, v, w are pairwise orthogonal vectors. Find the lengths of $u, v,$ and w and find orthonormal-vectors u_1, v_1, w_1 from vectors u, v, w.

Soln: → i) $u^T v = [1 \ 2 \ 2] \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$= 2 - 4 + 2 = 0$

$v^T w = [2 \ -2 \ 1] \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$= 4 - 2 - 2 = 0$

and $w^T u = [2 \ 1 \ -2] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$= 2 + 2 - 4 = 0$

Hence $(u, v) = (v, w) = (w, u) = 0$, means that u, v, w are pairwise orthogonal in R^3.

ii) length of $u = \|u\| = (u_1^2 + u_2^2 + u_3^2)^{1/2}$
$= (1 + 4 + 4)^{1/2} = 3$

length of $v = \|v\| = (v_1^2 + v_2^2 + v_3^2)^{1/2}$
$= (4 + 4 + 1)^{1/2} = 3$

and the length of $w = \|w\| = (w_1^2 + w_2^2 + w_3^2)^{1/2}$
$= (4 + 1 + 4)^{1/2} = 3$.

∴ Lengths of $u, v,$ and w are 3, 3, 3 respectively

(PTO)

$$\therefore u_1 = \frac{u}{\|u\|} = \frac{\begin{bmatrix}1\\2\\2\end{bmatrix}}{3} = \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$$

$$v_1 = \frac{v}{\|v\|} = \frac{\begin{bmatrix}2\\-2\\1\end{bmatrix}}{3} = \begin{bmatrix}2/3\\-2/3\\1/3\end{bmatrix}$$

and $$w_1 = \frac{w}{\|w\|} = \frac{\begin{bmatrix}2\\1\\-2\end{bmatrix}}{3} = \begin{bmatrix}2/3\\1/3\\-2/3\end{bmatrix}$$

$$\|u_1\| = (u_1 \cdot u_1)^{1/2} = (u_1^T \cdot u)^{1/2}$$

$$= \left[\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{bmatrix}\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}\right]^{1/2} = \left(\frac{1}{9} + \frac{4}{9} + \frac{4}{9}\right)^{1/2}$$

$$= (1)^{1/2} \left\{= \frac{1+4+4}{9} = \frac{9}{9} = 1\right)$$

similarly $\|v_1\| = 1$ and $\|w_1\| = 1$

Hence u_1, v_1 and w_1 ~~are~~ form a set of orthonormal vectors //

Q 9 a. (i) Reduce the matrix A to U and find the det(A) using pivots of A. (06 Marks)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

(ii) By applying row-operations to produce an upper triangular matrix U, compute the det(A)

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$$

Ans:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{bmatrix}$$

$R_3 \rightarrow R_3 - \frac{3R_2}{2}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -3/2 \end{bmatrix} \Rightarrow U.$$

$\therefore |A| = (1)(-2)(-3/2) = 3$ //.

PTO)

ii) By applying row row operations to produce an upper triangular matrix, to compute det (A)

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 3 & 7 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$R_4 \rightarrow R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$\therefore \; |A| = 1 \times 2 \times 3 \times 6 = 36$ //.

Q 9 b. Find the eigen values & eigen-vectors of a matrix A. (06 Marks)

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

Soln:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$[A - \lambda I] = 0$$

$$\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda = 5, -1$$

$\therefore$ eigen values are $\lambda = 5$ and $\lambda = -1$.

Case I: when $\lambda = 5$

$$-4x + 4y = 0$$

$$4x = 4y$$

$$\frac{x}{4} = \frac{y}{4}$$

$$x_1 = [1, 1]^T$$

Case II: when $\lambda = -1$

$$2x + 4y = 0$$

$$2x = -4y$$

$$x = -2y$$

$$\frac{x}{-2} = \frac{y}{1}$$

$$x_2 = [-2, 1]^T.$$

x_1 and x_2 are eigen vectors//.

Q 9 c. Factor the matrix A into $A = X \Lambda X^{-1}$ using diagonalization and hence find A^3.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$ (08 marks)

soln:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$[A - \lambda I] = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(3-\lambda) = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 1, 3.$$

$\therefore$ Eigen values are $\lambda = 1$ and $\lambda = 3$.

Case I: $\lambda = 1$,

$$0x + 2y = 0$$

$$0x = -2y$$

$$\frac{x}{2} = -\frac{y}{0}$$

$$\therefore x_1 = [2, 0]^T$$

Case II: $\lambda = 3$.

$$-2x + 2y = 0$$

$$2x = 2y$$

$$\frac{x}{2} = \frac{y}{2}$$

$$x_2 = [1, 1]^T$$

$$\therefore X = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ diagonal matrix

$$A = X \Lambda X^{-1} \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\therefore A^3 = X \Lambda^3 X^{-1} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$$ Ans

Q 10 a. Factorize the matrix A into $A = U\Sigma V^T$ using SVD. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. (08 marks)

Soln:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$[A^T A - \lambda I] = 0$$

$$\lambda^3 - (\Sigma d)\lambda^2 + \Sigma m(d)\lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 3\lambda = 0$$

$\lambda = 1, 3, 0$ $\quad \sigma_1 \sqrt{1}$, $\sigma_2 = \sqrt{3}$

$$V = \frac{[2, -1, 2]^T}{\sqrt{4+1+4}}$$

$$V = \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$$

Case ii) $\lambda = -3$

$$4x + 0y + 2z = 0$$

$$0x + 4y - 2z = 0$$

$$\frac{x}{\begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 4 & 2 \\ 0 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix}}$$

$$\frac{x}{0-4} = \frac{-y}{-8} = \frac{z}{8}$$

$X_2 = [-4, 8, 8]$, $\therefore X_2 = [-1, 2, 2]$

$$V = \frac{+[-1, 2, 2]^T}{\sqrt{9}}$$

$$V = -\frac{1}{\sqrt{3}}, \frac{2}{3}, \frac{2}{3}$$

Case iii) $\lambda = 0$

$$1x - 0y + 2z = 0$$

$$0x - 1y - 2z = 0$$

$$\frac{x}{\begin{vmatrix} 0 & 2 \\ -1 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{-1}$$

$$X_3 = [2, 2, -1]$$

$$V = \frac{[2, 2, -1]}{\sqrt{9}} = \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$$

$$Q = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$$

For orthogonal vectors

$$X_1 X_2^T = X_2 X_3^T = X_3 X_1^T = 0$$

$$[2, -1, 2] \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 0$$

$\therefore$ Eigen vectors are orthogonal for symmetric matrix.

Q 10 b. i> What is a positive definite matrix ? mention the methods of testing positive definiteness.

ii> Check the following ~~for~~ matrix for positive definiteness.

$$S = \begin{pmatrix} 5 & 6 \\ 6 & 7 \end{pmatrix}$$

(06 Marks)

Ans: → Symmetric matrix with positive eigen-values are called " +ve definite matrices.

Testing methods are

i> All $\lambda > 0$

ii> All pivots ellimination > 0

iii> All upper left determinants > 0

iv> $S = A^T A$ when S has independent colns

v> Energy test $x^T S x > 0$

ii> Checking the matrix for positive definiteness.

$$S_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$$

I det $= 5 > 0$

II det $= 35 - 36$

III det $= -1 < 0$

∴ S_1 is not a positive definite.

Q 10 c. Find the orthogonal matrix Q that diagonalizes the following symmetric matrix.

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

(06 Marks)

(PTO)

SVD continued ----

Soln:

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\lambda^3 + 0\lambda^2 + \left\{ \begin{vmatrix} -1 & -2 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \right\} \lambda - 0 = 0$$

$$\lambda^3 + 0\lambda^2 + \{ (0-4) + (0-4) + (-1) \} \lambda = 0$$

$$\lambda^3 + \{-4-4-1\}\lambda = 0$$

$$\lambda^3 - 9\lambda = 0$$

$$\lambda = 3, -3, 0$$

$$\left. \begin{aligned} (1-\lambda)x + 0y + 2z &= 0 \\ 0x - (-1-\lambda)y - 2z &= 0 \\ 2x - 2y + (0-\lambda)z &= 0 \end{aligned} \right\} \rightarrow \text{\textcircled{}}$$

Case I: $\lambda = 3$

$$-2x + 0y + 2z = 0$$

$$0x - (4y) - 2z = 0$$

$$\frac{x}{\begin{vmatrix} 0 & 2 \\ -4 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & 2 \\ 0 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix}}$$

$$\frac{x}{8} = \frac{-y}{4} = \frac{z}{8}$$

$$X_1 = [2, -1, 2]^T$$

$$[A^TA - \lambda I][x] = 0$$

$$\left. \begin{aligned} (1-\lambda)x + y + 0z &= 0 \\ 1x + (2-\lambda)y + 1z &= 0 \\ 0x + y + (1-\lambda)z &= 0 \end{aligned} \right\}$$

Case i) $\lambda = 1$

$$0x + y + 0z = 0$$

$$x + y + z = 0$$

Case ii) $\lambda = 3$

$$-2x + y + 0z = 0$$

$$x - y + z = 0$$

10 c continued....

$$\frac{x}{1-0} = \frac{-y}{0} = \frac{z}{-1}$$

$$x_1 = \frac{v_1}{|v_1|}$$

$$x_1 = \left[\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right]$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$$x_2 = \frac{v_2}{|v_2|}$$

$$x_2 = \left[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$$

Case iii) $\lambda = 0$

$$x + y + 0z = 0$$

$$x + 2y + z = 0$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$$v_3 = [1, -1, 1]^T$$

$$\therefore x_3 = \frac{v_3}{\|v_3\|} = \left[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T$$

$$V = [x_1, x_2, x_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -\frac{1}{\sqrt{2}} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A x_1$$

$$= \frac{1}{\sqrt{1}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A x_2$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

(P.T.O.)

$$U = [u_1 \; u_2]$$

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$\therefore A = U \Sigma V^T$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{\ } \\ 0 & 2/\sqrt{6} & -1/ \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} //$$