AY: 2020-21

Department: MATHEMATICS
Subject with Sub. Code: Additional Mathematics-2/18MATDIP41
Semester / Division: 4 **Name of Faculty: Prof Hazel S Mathias**

Q.No.	Solution and Scheme	
1. a)	Find the rank of the following matrix by applying elementary row transformations. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$	
Ans:	$R_1 \leftrightarrow R_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$	
	$R_3 \to -3R_1 + R_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$	
	$R_3 \to -R_2 + R_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	
	A matrix A in the row echelon form has 2 nonzero rows. Thus $\rho(A) = 2$.	
1. b)	Solve the following system of linear equations by Gauss elimination method:	
	$x + 2y + z = 3$; $2x + 3y + 3z = 10$; $3x - y + 2z = 13$	
Ans:	Augmented matrix $[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$	
	$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ $\quad [A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right]$	

Q.No	Solution and Scheme
	$R_3 \to R_3 - 7R_2$ $[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 1 & : & 4 \\ 0 & 0 & -8 & : & -24 \end{bmatrix}$
	Hence we've $x+2y+z=3$ $-y+z=4$ $-8z=-24 \Rightarrow z=3$
	By back substitution we've $y=-1$ and $x=2$
	Thus $x=2, y=-1, z=3$ is the required solution.
1.c)	Find all the eigenvalues and eigen vector corresponding to the smallest eigen value of $A=\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$
Ans:-	The characteristic equation of A is $\lvert A-\lambda I\rvert=0$.
	i.e $\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix}=0$
	On expanding we've $(2-\lambda)\left[(2-\lambda)^2-0\right]-0\left[0(2-\lambda)-0\right]-1\left[0-(-1)(2-\lambda)\right]=0$
	i.e $(2-\lambda)^3-(2-\lambda)=0$. $8-\lambda^3-12\lambda+6\lambda^2-2+\lambda=0$ i.e $\lambda^3-6\lambda^2+11\lambda-6=0$
	$\lambda=1$ is the root by inspection.
	Now by synthetic division $\begin{array}{c\|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \underline{0} \end{array}$
	$\therefore \lambda^2-5\lambda+6=0 \Rightarrow \lambda=2,3$
	$\therefore \lambda=1,2,3$ are the eigen values, and the smallest eigen value is $\lambda=1$. we shall find the eigen vector

Q.No	Solution
	Corresponding to $\lambda = 1$.
	Now we form the system of equations:
	$(2-\lambda)x - z = 0$; $(2-\lambda)y = 0$; $-x + (2-\lambda)z = 0$
	Let $\lambda = 1$ we get $x - z = 0$; $y = 0$; $-x + z = 0$
	Applying the rule of cross multiplication for the first 2 equations we get,
	$\dfrac{x}{\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}} = \dfrac{-y}{\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}$ i.e $\dfrac{x}{1} = \dfrac{-y}{0} = \dfrac{z}{1}$
	$\therefore x = 1,\ y = 0,\ z = 1$
	$\therefore [1\ \ 0\ \ 1]^T$ is the eigen vector corresponding to $\lambda = 1$.
OR	
2.a)	Reduce the matrix into its echelon form, and hence find its rank. $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$
Ans:-	$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ $\quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix}$
	$R_3 \to R_3 - R_2$ $\quad A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$
	The matrix A in its row echelon form has one non zero row. $\therefore \rho(A) = 2$.
2 b)	Find all the eigen values and eigen vector corresponding to the largest eigen value of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Q.No	Solution and Scheme	
Ans.	The characteristic equation of A is $\|A-\lambda I\|=0$	

i.e $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

On expanding we get,

$(3-\lambda)\left[(5-\lambda)(3-\lambda)-1\right]-(-1)\left[-(3-\lambda)-(-1)\right]+1\left[1-(5-\lambda)\right]=0$

$(3-\lambda)(14+\lambda^2-8\lambda)-6+2\lambda=0$

i.e $\lambda^3-11\lambda^2+36\lambda-36=0$

$\lambda=2$ is the root by inspection. Now by synthetic division we get,

$$\begin{array}{c|cccc} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\therefore\ \lambda^2-9\lambda+18=0$ i.e $\lambda=3,6$

$\therefore\ \lambda=2,3,6$ are the eigen values and the largest eigen value is $\lambda=6$. Now we shall find the eigen vector corresponding to $\lambda=6$.

We form the equations; $(3-\lambda)x-y+z=0$

$-x+(5-\lambda)y-z=0$

$x-y+(3-\lambda)z=0$

Let $\lambda=6$, we get

$-3x-y+z=0$ — ①

$-x-y-z=0$ — ②

$x-y+6z=0$ — ③

Applying the rule of cross multiplication to ① & ②

$$\frac{x}{\begin{vmatrix}-1 & 1\\ -1 & -1\end{vmatrix}} = \frac{-y}{\begin{vmatrix}-3 & 1\\ -1 & -1\end{vmatrix}} = \frac{z}{\begin{vmatrix}-3 & -1\\ -1 & 1\end{vmatrix}} \quad \text{i.e} \quad \frac{x}{2}=\frac{-y}{4}=\frac{z}{2}$$

$\therefore\ X_1=[2\ \ -4\ \ 2]^T$ is the eigen vector corresponding to the largest eigen value $\lambda=6$.

Q.No	Solution
2 c)	Solve the system of linear equations $2x+y+z=7$; $x+3y+z=10$; $x+y+z=15$ by applying Gauss elimination method.

Ans:- Augmented Matrix $[A:B] = \left[\begin{array}{ccc:c} 2 & 1 & 1 & 7 \\ 1 & 3 & 1 & 10 \\ 1 & 1 & 1 & 15 \end{array}\right]$

$R_1 \leftrightarrow R_2$ $[A:B] \sim \left[\begin{array}{ccc:c} 1 & 3 & 1 & 10 \\ 2 & 1 & 1 & 7 \\ 1 & 1 & 1 & 15 \end{array}\right]$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$ $[A:B] \sim \left[\begin{array}{ccc:c} 1 & 3 & 1 & 10 \\ 0 & -5 & -1 & -13 \\ 0 & -2 & 0 & 5 \end{array}\right]$

$R_3 \to 5R_3 - 2R_2$ $[A:B] \sim \left[\begin{array}{ccc:c} 1 & 3 & 1 & 10 \\ 0 & -5 & -1 & -13 \\ 0 & 0 & 2 & 51 \end{array}\right]$

Now we form the system of equations:

$x+3y+z=10$; $-5y-z=-13$; $2z=51$

$2z=51 \Rightarrow z=\frac{51}{2}$

$-5y-\frac{51}{2}=-13$ $\therefore y=-\frac{5}{2}$

$x+3\left(-\frac{5}{2}\right)+\frac{51}{2}=10$ $\therefore x=-8$

Thus $x=-8$, $y=-\frac{5}{2}$, $z=\frac{51}{2}$ is the required solution

3a)	Find a real root of $xe^x-\cos x=0$, correct to 3 decimal places lying in $(0.5, 0.6)$. By using Regula-Falsi Method.

Ans:- Let $f(x)=xe^x-\cos x$

1st iteration: $a=0.5$ $f(a)=-0.0532$

$b=0.6$ $f(b)=0.2679$

Q.No	Solution and Scheme

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{0.5(0.2679) - (0.6)(-0.0532)}{0.2679 - (-0.0532)}$$

$$x_1 = 0.5166$$

2nd iteration: $f(x_1) = f(0.5166) = -0.0035 < 0$

$\therefore$ the root lies in $(0.5166, 0.6)$

$a = 0.5166 \quad \therefore f(a) = -0.0035$

$b = 0.6 \quad f(b) = 0.2679$

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{(0.5166)(0.2679) - (0.6)(-0.0035)}{0.2679 - (-0.0035)}$$

$\therefore x_2 = 0.5177$

3rd iteration: $f(x_2) = f(0.5177) = -0.00017 < 0$

$\therefore$ the root lies in $(0.5177, 0.6)$

$a = 0.5177 \quad \therefore f(a) = -0.00017$

$b = 0.6 \quad \therefore f(b) = 0.2679$

$$x_3 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{(0.5177)(0.2679) - (0.6)(-0.00017)}{0.2679 - (-0.00017)}$$

$\therefore x_3 = 0.5178$

Hence the real root correct to 3 decimal places is 0.517

3b) Use an appropriate interpolation formula to compute $f(6)$ using the following data

x	1	2	3	4	5
$f(x)$	1	−1	1	−1	1

Ans: We shall use Newton's backward interpolation formula:

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

Q.No	Solution

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
		-2			
2	-1		4		
		2		-8	
3	1		-4		16
		-2		8	
4	-1		4		
		2			
5	1				

$$r = \frac{x - x_n}{h} = \frac{6-5}{1} = 1$$

From the table $\nabla y_n = 2$, $\nabla^2 y_n = 4$, $\nabla^3 y_n = 8$, $\nabla^4 y_n = 16$

$$f(6) = 1 + 1.(2) + \frac{1(1+1)}{2!} \times 4 + \frac{1(1+1)(1+2)}{3!} \times 8 + \frac{1(1+1)(1+2)(1+3)}{4!} \times 16$$

$$= 31$$

3c) Evaluate $\int_0^{\pi/2} \sqrt{\sin x}\, dx$ using simpson's $(1/3)^{rd}$ rule, taking 10 equal parts.

Ans:- Length of each part $h = \frac{\pi/2 - 0}{10} = \frac{\pi}{20} = 9^\circ$, $n = 10$

x°	0°	9°	18°	27°	36°	45°	54°	63°	72°	81°	90°
$y = \sqrt{\sin x}$	0	0.3955	0.5558	0.6737	0.7666	0.8408	0.8994	0.9439	0.9752	0.9938	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Simpson's $1/3^{rd}$ rule for $n=10$ is given by,

$$\int_a^b y\,dx = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

Q.No	Solution and Scheme
	$\int_0^{\pi/2}\sqrt{\sin x}\,dx = \frac{1}{3}\cdot\frac{\pi}{20}\left[(0+1)+4(0.3955+0.6737+0.8408+0.9439+0.9938)+2(0.558+0.7666+0.8994+0.9752)\right]$
	$=1.1873$
4.a)	Find a real root of the equation $x\sin x+\cos x=0$, near $x=\pi$ correct to 4 decimal places, using Newton Raphson method.
Ans:.	Let $f(x)=x\sin x+\cos x$
	$f'(x)=x\cos x+\sin x-\sin x = x\cos x$, $x_0=\pi$
	By Newton Raphson method $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$
	1st iteration: $x_1=x_0-\frac{f(x_0)}{f'(x_0)} = \pi-\frac{f(\pi)}{f'(\pi)} = \pi-\frac{1}{\pi}$
	$\therefore x_1=2.8233$
	2nd iteration: $x_2=x_1-\frac{f(x_1)}{f'(x_1)} = 2.8233-\frac{f(2.8233)}{f'(2.8233)}$
	$\therefore x_2=2.8233-\frac{[2.8233\sin(2.8233)+\cos(2.8233)]}{2.8233\cos(2.8233)}$
	$x_2=2.7986.$
	3rd iteration: $x_3=x_2-\frac{f(x_2)}{f'(x_2)}$
	$\therefore x_3=2.7986-\frac{[2.7986\sin(2.7986)+\cos(2.7986)]}{2.7986\cos(2.7986)}$
	$\therefore x_3=2.7984.$
	4th iteration: $x_3-\frac{f(x_3)}{f'(x_3)}=x_4$
	$\therefore x_4=2.7984-\frac{f(2.7984)}{f'(2.7984)}=2.7984$
	Thus the required real root is 2.7984.

Q.No	Solution

4b) Use an appropriate interpolation formula to compute $f(0.15)$ using the following data.

x	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	2.68	3.04	3.38	3.68	3.96	4.21

Ans:- The value $x=0.15$ is nearer to $x=0.1$ and hence Newton's forward interpolation formula is appropriate. We shall construct the forward difference table:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.1	2.68					
		0.66				
0.2	3.04		−0.32			
		0.34		0.28		
0.3	3.38		−0.04		−0.26	
		0.3		0.02		0.23
0.4	3.68		−0.02		−0.03	
		0.28		−0.01		
0.5	3.96		−0.03			
		0.25				
0.6	4.21					

We've Newton's forward interpolation formula:

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

where $r = \dfrac{x-x_0}{h} = \dfrac{0.15-0.1}{0.1} = 0.5$.

$$y_r = 2.68 + 0.5(0.66) + \frac{0.5(0.5-1)}{2!}(-0.32) + \frac{0.5(0.5-1)(0.5-2)}{3!}(0.28) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(-0.26) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)(0.5-4)}{5!} \times 0.23$$

Q.No	Solution and Scheme	
	$\therefore f(0.15) = 2.89$	

4 c) Use Weddle's Rule to evaluate $\int_0^1 \frac{x\,dx}{1+x^2}$, by taking seven ordinates.

Ans:- Seven ordinates means that $[0,1]$ must be divided into 6 equal parts.

Length of each part $(h) = \frac{1-0}{6} = \frac{1}{6}$; $n=6$.

The points of division are $x = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$

The corresponding values of $y = x/1+x^2$ are computed.

$x=0, \quad y = \frac{0}{1+0^2} = 0 \; (y_0)$

$x = \frac{1}{6}, \quad y = \frac{1/6}{1+(1/6)^2} = \frac{6}{37} \; (y_1)$

$x = \frac{1}{3}, \quad y = \frac{1/3}{1+(1/3)^2} = \frac{3}{10} \; (y_2)$

$x = \frac{1}{2}, \quad y = \frac{1/2}{1+(1/2)^2} = \frac{2}{5} \; (y_3)$

$x = \frac{2}{3}, \quad y = \frac{2/3}{1+(2/3)^2} = \frac{6}{13} \; (y_4)$

$x = \frac{5}{6}, \quad y = \frac{5/6}{1+(5/6)^2} = \frac{30}{61} \; (y_5)$

$x = 1, \quad y = \frac{1}{1+1^2} = \frac{1}{2} \; (y_6)$

Weddle's Rule for $n=6$ is given by,

$$\int_a^b y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 \right]$$

$$= \frac{3}{10} \times \frac{1}{6}\left[0 + 5\left(\frac{6}{37}\right) + \frac{3}{10} + 6\left(\frac{2}{5}\right) + \frac{6}{13} + 5\left(\frac{30}{61}\right) + \frac{1}{2} \right]$$

Thus $\int_0^1 \frac{x}{1+x^2}\,dx = 0.3466.$

Q.No	Solution and Scheme	
	Module - 03	
5 a)	Solve: $(D^3 - D^2 + 4D - 4)\,y = 0$	
Ans:	AE is $f(m) = 0$ i.e $m^3 - m^2 + 4m - 4 = 0$ i.e $m^2(m-1) + 4(m-1) = 0$ i.e $(m-1)(m^2+4) = 0 \Rightarrow m = 1, \pm 2i$ $\therefore y_c = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x$	
5 b)	Solve: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2\cosh x$	
Ans:	we've $(D^2+4D+5)\,y = 2\cosh x$ AE is $f(m)=0$ i.e $m^2+4m+5=0$. Hence $m = \frac{-4 \pm \sqrt{16-4(1)(5)}}{2(1)} = \frac{-4 \pm 2i}{2} = -2 \pm i$ $\therefore y_c = e^{-2x}(c_1\cos x + c_2 \sin x)$ $\therefore y_p = \frac{2\cosh x}{D^2+4D+5}$ $\quad \cosh x = \frac{e^x + e^{-x}}{2}$ $\therefore y_p = \frac{e^x + e^{-x}}{D^2+4D+5} = \frac{e^x}{D^2+4D+5} + \frac{e^{-x}}{D^2+4D+5}$ $y_p = \frac{e^x}{1^2+4(1)+5} + \frac{e^{-x}}{(-1)^2+4(-1)+5} = \frac{e^x}{10} + \frac{e^{-x}}{2}$ Complete solution: $y = y_c + y_p$ That is $y = e^{-2x}(c_1\cos x + c_2 \sin x) + \frac{e^x}{10} + \frac{e^{-x}}{2}$	
5c)	Solve: $y'' - 4y = \cos 2x$	
Ans:	we've $(D^2 - 4)\,y = \cos 2x$ AE is $f(m) = 0$ i.e $m^2 - 4 = 0$ $\therefore m = \pm 2$	

Q.No	Solution

The roots are real and distinct.

$\therefore y_c = c_1 e^{2x} + c_2 e^{-2x}$.

$y_p = \dfrac{\phi(x)}{f(D)} = \dfrac{\cos 2x}{D^2 - 4}$ $\quad a = 2$, Replacing D^2 by $-2^2 = -4$

$y_p = \dfrac{\cos 2x}{-4-4} = \dfrac{\cos 2x}{-8}$

$\therefore$ The complete solution $y = y_c + y_p$

$\therefore y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{\cos 2x}{8}$

OR

6 a) Solve: $(D^4 + 4D^3 - 5D^2 - 36D - 36)y = 0$

Ans:- AE is $f(m) = 0$ i.e $m^4 + 4m^3 + 5m^2 - 36m - 36 = 0$

$m = -2$ is the root by inspection

$$\begin{array}{r|rrrrr} -2 & 1 & 4 & -5 & -36 & -36 \\ & 0 & -2 & -4 & 18 & 36 \\ \hline & 1 & 2 & -9 & -18 & \boxed{0} \end{array}$$

i.e $m^3 + 2m^2 - 9m - 18 = 0$

Again $m = -2$ is the root by inspection

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -9 & -18 \\ & 0 & -2 & 0 & 18 \\ \hline & 1 & 0 & -9 & \boxed{0} \end{array}$$

i.e $m^2 - 9 = 0 \Rightarrow m = \pm 3$

$\therefore m = -2, -2, +3, -3$ are the roots.

$\therefore y_c = (c_1 + c_2 x) e^{-x} + c_3 e^{3x} + c_4 e^{-3x}$

Q.No	Solution
6 b)	Solve: $y'' + 5y' + 6y = e^{-2x}$
Ans:	AE is $f(m)=0$ i.e $m^2+5m+6=0$ $\therefore m = -2, -3$ roots are real and distinct $\therefore y_c = c_1 e^{-2x} + c_2 e^{-3x}$ $\therefore y_p = \dfrac{\phi(x)}{f(D)} = \dfrac{e^{-2x}}{D^2+5D+6}$, Replacing D by -2. $y_p = \dfrac{e^{-2x}}{(-2)^2+5(-2)+6}$ (Dr. = 0) $y_p = x \cdot \dfrac{e^{-2x}}{2D+5}$, Replacing D by -2 $\therefore y_p = x \cdot \dfrac{e^{-2x}}{2(-2)+5} = x e^{-2x}$ $\therefore y = y_c + y_p$, the complete solution $\therefore y = c_1 e^{-2x} + c_2 e^{-3x} + x \cdot e^{-2x}$ is the solution.
6 c)	$(D^2+5D+6)y = \sin x$
Ans:	AE is $f(m)=0$ i.e $m^2+5m+6=0$ $\therefore m=-2,-3$, roots are real and distinct $\therefore y_c = c_1 e^{-2x} + c_2 e^{-3x}$ $y_p = \dfrac{\phi(x)}{f(D)} = \dfrac{\sin x}{D^2+5D+6}$, Replacing D by $-1^2=-1$ $y_p = \dfrac{\sin x}{5D+5} = \dfrac{1}{5} \cdot \dfrac{\sin x}{D+1} \times \dfrac{D-1}{D-1}$ $= \dfrac{1}{5} \cdot \dfrac{(D-1)\sin x}{D^2-1}$, Replacing D^2 by -1 $y_p = \dfrac{1}{5}\left[\dfrac{\cos x - \sin x}{-2}\right] = \dfrac{\sin x - \cos x}{10}$ Complete solution: $y = y_c + y_p = c_1 e^{-2x} + c_2 e^{-3x} + \dfrac{\sin x - \cos x}{10}$

Q.No	Solution and Scheme
	Module - 04
7.a)	Form the partial differential equation by eliminating the arbitrary function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$
Ans:-	By data $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ ——— ①
	Differentiating ① partially w.r.t. x and y we've,
	$\frac{\partial z}{\partial x} = p = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$
	$\frac{\partial z}{\partial y} = q = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$
	i.e $px^2 = -2f'\left(\frac{1}{x} + \log y\right)$ ——— ②
	and $(q-2y)\,y = 2f'\left(\frac{1}{x} + \log y\right)$ ——— ③
	Now dividing (2) by (3) we've
	$\frac{px^2}{(q-2y)\,y} = -1$ OR $px^2 = -qy + 2y^2$
	Thus $px^2 + qy = 2y^2$ is the required PDE.
7.b)	Form the partial differential equation by eliminating the arbitrary constants from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
Ans:-	By data $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ——— ①
	Differentiating (1) partially w.r.t. x and y,
	$\frac{2x}{a^2} + \frac{2z}{c^2}p = 0$ and $\frac{2y}{b^2} + \frac{2z}{c^2}q = 0$
	i.e $\frac{x}{a^2} + \frac{zp}{c^2} = 0$ ——— ②
	$\frac{y}{b^2} + \frac{zq}{c^2} = 0$ ——— ③
	Since there are 3 arbitrary constants, we differentiate further.

Q.No	Solution

Differentiating (2) w.r.t. x partially again we get,

$$\frac{1}{a^2} + \frac{1}{c^2}(zr + p^2) = 0 \quad \text{———} \quad (4)$$

$$\because \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = r$$

Now from (2), $\frac{x}{a^2} = \frac{-zp}{c^2}$ OR $\frac{1}{a^2} = \frac{-zp}{c^2 x}$

Substituting this in (4) we get,

$$-\frac{zp}{c^2 x} = -\frac{1}{c^2}(zr + p^2) \quad \text{OR} \quad zp = x(zr + p^2)$$

Thus $z \cdot \frac{\partial z}{\partial x} = xz\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2$ is the required PDE.

7.c) Solve $\frac{\partial^2 z}{\partial y^2} = z$, given that when $y = 0$, $z = e^x$ & $\frac{\partial z}{\partial y} = e^{-x}$

Ans: Let us suppose that z is a function of y only.

The given PDE assumes the form of ODE,

$$\frac{d^2 z}{dy^2} = z \quad \text{OR} \quad \frac{d^2 z}{dy^2} - z = 0 \quad \text{OR} \quad (D^2 - 1)z = 0 \quad \text{where } D = \frac{d}{dy}$$

AE is $m^2 - 1 = 0$ $\therefore m = \pm 1$ (real and distinct roots)

The solution of ODE is $z = c_1 e^y + c_2 e^{-y}$.

Solution of the PDE is got by replacing c_1 and c_2 by functions of x. Hence the solution of the PDE is,

$$z = f(x)e^y + g(x)\cdot e^{-y} \quad \text{———} \quad (1)$$

Now we shall apply the given conditions to find $f(x)$ and $g(x)$. By data, when $y = 0$, $z = e^x$. Hence (1) becomes,

$$e^x = f(x) + g(x)$$

Also by data, when $y = 0$, $\frac{\partial z}{\partial y} = e^{-x}$

Differentiating (1) w.r.t. y partially we get,

$$\frac{\partial z}{\partial y} = f(x)\cdot e^y - g(x)\cdot e^{-y}$$

Q.No	Solution
	Applying the condition we get, $e^{-x} = f(x) - g(x)$
	Now we shall solve, $f(x) + g(x) = e^{x}$ and $f(x) - g(x) = e^{-x}$
	Adding and subtracting these equations we get,
	$2f(x) = e^{x} + e^{-x}$ OR $f(x) = \frac{e^{x}+e^{-x}}{2} = \cosh x$
	$2g(x) = e^{x} - e^{-x}$ OR $g(x) = \frac{e^{x}-e^{-x}}{2} = \sinh x$
	We shall substitute these in ①
	Thus $z = \cosh x\, e^{y} + \sinh x\, e^{-y}$ is the required solution
	OR
8a)	Form the partial differential equation by eliminating the arbitrary function from $f(x^2+y^2, z-xy)=0$.
Ans:-	We've by data, $f(u,v)=0$ where $u = x^2+y^2$ and $v = z-xy$ — ①
	Now, $\frac{\partial u}{\partial x} = 2x$; $\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} - y = p - y$
	$\frac{\partial u}{\partial y} = 2y$; $\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} - x = q - x$
	Let us differentiate ① partially w.r.t. x and y by applying chain rule.
	$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = 0$ OR $\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} = -\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$ — ②
	$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = 0$ OR $\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} = -\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$ — ③
	Dividing ② by ③ we've $\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y}$
	That is $\frac{2x}{2y} = \frac{p-y}{q-x}$ OR $\frac{x}{y} = \frac{p-y}{q-x}$
	OR $y(p-y) = x(q-x)$ OR $yp - y^2 = xq - x^2$
	Thus $yp - xq = y^2 - x^2$ is the required PDE.
8b)	Form the partial differential equation by eliminating the arbitrary functions from $z = y f(x) + x\phi(y)$
Ans:-	By data, $z = y f(x) + x\phi(y)$ — ①

Q.No	Solution
	Differentiating w.r.t x and y partially,
	$\frac{\partial z}{\partial x} = p = y f'(x) + \phi(y)$ ——— ②
	$\frac{\partial z}{\partial y} = q = f(x) + x\phi'(y)$ ——— ③
	$\frac{\partial^2 z}{\partial x^2} = r = y f''(x)$ ——— ④
	$\frac{\partial^2 z}{\partial x \partial y} = s = f'(x) + \phi'(y)$ ——— ⑤
	$\frac{\partial^2 z}{\partial y^2} = t = x\phi''(y)$ ——— ⑥
	Now from ②, $\frac{p-\phi(y)}{y} = f'(x)$, from ③, $\frac{q-f(x)}{x} = \phi'(y)$
	Using these in ⑤ we get,
	$s = \frac{p-\phi(y)}{y} + \frac{q-f(x)}{x}$ OR $s = \frac{px - x\phi(y) + qy - yf(x)}{xy}$
	$\therefore xys = px + qy - [x\phi(y) + yf(x)]$
	Using ① in RHS we get,
	$xys = px + qy - z$ OR $xys + z = px + qy$
	Thus $xy\frac{\partial^2 z}{\partial x \partial y} + z = x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y}$ is the PDE.
8. c)	Solve $\frac{\partial^2 z}{\partial x \partial y} = e^{-2y}\cos 3x$, for which $\frac{\partial z}{\partial y} = 0$ when $x = 0$ and $z = 0$ when $y = 0$.
Ans:-	Since the condition is in terms of $\frac{\partial z}{\partial y}$ and write the given PDE as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^{-2y}\cos 3x$
	Integrating w.r.t. x treating y as constant,
	$\frac{\partial z}{\partial y} = e^{-2y} \cdot \frac{\sin 3x}{3} + g(y)$ ——— ①
	Integrating w.r.t. y treating x as constant,
	$z = \frac{\sin 3x}{3} \cdot \frac{e^{-2y}}{-2} + \int g(y)\,dy + f(x)$

Q.No	Solution
	i.e $z = \frac{\sin 3x \cdot e^{-2y}}{-6} + G(y) + f(x)$, $G(y) = \int g(y)\,dy$ — ②
	Also by data, $\frac{\partial z}{\partial y} = 0$ when $x=0$. Using this in ①
	$g(y) = 0 \Rightarrow G(y) = 0$ and we substitute this in ②
	$\therefore z = \frac{\sin 3x \cdot e^{-2y}}{-6} + 0 + f(x)$ ——— ③
	Also by data $z=0$ when $y=0$. Using this in ③,
	$0 = \frac{\sin 3x}{-6} + f(x) \Rightarrow f(x) = \frac{\sin 3x}{6}$
	Thus the solution is given by,
	$z = \frac{\sin 3x \cdot e^{-2y}}{-6} + \frac{\sin 3x}{6} = \frac{\sin 3x}{6}(1 - e^{-2y})$
	Module - 05
9.a)	Define conditional probability. For any 2 events A & B, prove that $P(A \cap B) = P(A) \cdot P(B\|A)$
Ans:-	Let A and B be the two events. Probability of the happening of the event B when the event A has already happened is called the conditional probability denoted by $P(B\|A)$.
	$P(B\|A) = \frac{\text{Probability of the occurrence of both B and A}}{\text{Probability of the occurrence of the given event A}}$.
	i.e $P(B\|A) = \frac{P(A \cap B)}{P(A)}$ ——— ①
	Also $P(A\|B) = \frac{P(A \cap B)}{P(B)}$ ——— ②
	From ① $P(A \cap B) = P(A) \cdot P(B\|A)$ ——— ③.
	where $P(A) > 0$. This is called the multiplication rule of probability. If A and B are 2 independent events then $P(B\|A) = P(B)$.

Q.No	Solution
	Hence ③ becomes,
	$P(A\cap B) = P(A)\cdot P(B)$
	i.e $P(A\cap B) = P(A)\cdot P(B) \Leftrightarrow A$ and B are independent.
9.b)	The probability that 3 students A, B, C solve a problem is $1/2, 1/3, 1/4$ respectively. If the problem is simultaneously assigned to all of them, what is the probability that the problem is solved?.
Ans:-	P (problem is solved) $= 1 - P$ (Problem being unsolved by A, B, C)
	$P(E) + P(\bar{E}) = 1$
	$P(\bar{E})$ is the probability that the problem is not solved.
	$\therefore P(\bar{E}) = P(\bar{A})\cdot P(\bar{B})\cdot P(\bar{C}) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4} = \frac{1}{4}$
	Hence $P(E) = 1 - \frac{1}{4} = \frac{3}{4}$
	Thus $P(E) = 3/4$
9.c)	An office has 4 secretaries handing respectively 20%, 60%, 15% and 5% of the files of all government reports. The probability that they misfile such reports are respectively 0.05, 0.1, 0.1 and 0.05. Find the probability that the misfiled report can be blamed on the first secretary.
Ans:-	Let A_1, A_2, A_3, A_4 be the 4 secretaries of the office, respectively handling 20%, 60%, 15%, 5% of the files. Hence we've
	$P(A_1) = 20/100 = 0.2$, $P(A_2) = 0.6$ $P(A_3) = 0.15$, $P(A_4) = 0.05$
	Let E be the event of misfiling a report by the secretaries.
	$\therefore P(E\|A_1) = 0.05$, $P(E\|A_2) = 0.1$, $P(E\|A_3) = 0.1$, $P(E\|A_4) = 0.05$
	We need to find $P(A_1\|E)$ and we've by Baye's theorem

Q.No	Solution

$$P(A_1|E) = \frac{P(A_1)\,P(E|A_1)}{P(A_1)\,P(E|A_1) + P(A_2)\,P(E|A_2) + P(A_3)\,P(E|A_3) + P(A_4)\,P(E|A_4)}$$

$$= \frac{(0.2)\,(0.05)}{(0.2)\,(0.05) + (0.6)(0.1) + (0.15)(0.1) + (0.05)(0.05)}$$

Thus $P(A_1|E) = 0.1143$.

OR

10.a) State and Prove Baye's Theorem:

"Let $A_1, A_2, \ldots, A_n$ be a set of exhaustive and mutually exclusive events of the sample space S with $P(A_i) \neq 0$ for each i. If A is any other event associated with A_i, $(A \subset \bigcup_{i=1}^{n} A_i)$ with $P(A) \neq 0$ then

$$P(A_i|A) = \frac{P(A_i)\,P(A|A_i)}{\sum_{i=1}^{n} P(A_i)\,P(A|A_i)}$$

Proof:- we've $S = A_1 \cup A_2 \cup \ldots \cup A_n$ and $A \subset S$

$\therefore A = S \cap A = (A_1 \cup A_2 \cup \ldots \cup A_n) \cap A$

Using distributive law in the RHS we've

$A = (A_1 \cap A) \cup (A_2 \cap A) \cup \ldots \cup (A_n \cap A)$

Since $A_i \cap A$ for $i = 1$ to n are mutually exclusive, we've by applying the addition rule of probability,

$P(A) = P(A_1 \cap A) + P(A_2 \cap A) + \ldots + P(A_n \cap A)$

Now applying multiplication rule onto each term in the RHS we've $P(A) = P(A_1)\,P(A|A_1) + P(A_2)\,P(A|A_2) + \ldots + P(A_n)\,P(A|A_n)$

That is $P(A) = \sum_{i=1}^{n} P(A_i)\,P(A|A_i)$ ——— ①

The conditional probability of A_i for any i given A, is defined by $P(A_i|A) = \dfrac{P(A_i \cap A)}{P(A)} = \dfrac{P(A_i)\,P(A|A_i)}{P(A)}$

Using ① in the denominator of the RHS we've,

Q.No	Solution
	$P(A_i \mid A) = \dfrac{P(A_i)\,P(A \mid A_i)}{\sum_{i=1}^{n} P(A_i)\,P(A \mid A_i)}$
	This proves Baye's theorem for conditional probability.
10.b)	Three major parties A, B, C are contending for power in the elections of a state and the chance of their winning the election is in the ratio 1:3:5. The parties A, B, C respectively have probabilities of banning the online lottery 2/3, 1/3, 3/5. What is the probability that there will be a ban on the online lottery in the state? What is the probability that the ban is from party C?
Ans:-	$P(A) = 1/9$, $P(B) = 3/9 = 1/3$, $P(C) = 5/9$
	Let E be the event of banning the online lottery.
	$\therefore P(E \mid A) = 2/3$, $P(E \mid B) = 1/3$, $P(E \mid C) = 3/5$
	Hence $P(E) = P(A)P(E \mid A) + P(B)P(E \mid B) + P(C)P(E \mid C)$
	$P(E) = \frac{1}{9}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{5}{9}\cdot\frac{3}{5} = \frac{14}{27}$
	Now we shall find $P(C \mid E)$ and we're by Baye's theorem,
	$P(C \mid E) = \dfrac{P(C)\,P(E \mid C)}{P(E)} = \dfrac{5/9 \cdot 3/5}{14/27} = \dfrac{9}{14}$
	Thus the probability of ban from C is 9/14.
10.c)	A shooter can hit a target in 3 out of 4 shots and another shooter can hit the target in 2 out of 3 shots. Find the probability that the target is being hit (a) when both of them try b) by only one shooter.
Ans:-	Let S_1 and S_2 be the events that the shooters 1 and 2 hit the target.
	$\therefore P(S_1) = \frac{3}{4}$ and $P(S_2) = \frac{2}{3}$

Q.No	Solution	

a) $P(S_1 \cup S_2) = P(S_1) + P(S_2) - P(S_1) \cap S_2)$

but S_1 and S_2 are independent.

$\therefore P(S_1 \cup S_2) = P(S_1) + P(S_2) - P(S_1) \cdot P(S_2)$

$$= \frac{3}{4} + \frac{2}{3} - \frac{3}{4} \cdot \frac{2}{3} = \frac{11}{12}$$

Thus $P(S_1 \cup S_2) = 11/12$.

(b) Target being hit by only one shooter means,

$(S_1 \cap \bar{S}_2) \cup (\bar{S}_1 \cap S_2)$

$$P[(S_1 \cap \bar{S}_2) \cup (\bar{S}_1 \cap S_2)] = P(S_1 \cap \bar{S}_2) + P(\bar{S}_1 \cap S_2)$$

$$= P(S_1) \cdot P(\bar{S}_2) + P(\bar{S}_1) \cdot P(S_2)$$

$$= \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} = \frac{5}{12}$$

Thus the required probability is $5/12$.