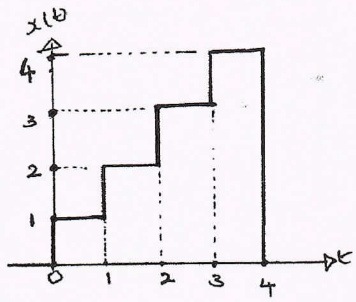
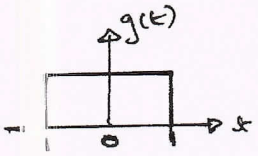
Model Question Paper-1 with effect from 2019-20 (CBCS Scheme)

USN | | | | | | | | | | |

Fourth Semester B.E. Degree Examination
Signals and Systems

TIME: 03 Hours **Max. Marks: 100**

Note: 01. Answer any **FIVE** full questions, choosing at least **ONE** question from each **MODULE**.
02. Short forms used take usual meaning.
03. Missing data may be suitably assumed

		Module -1	***Bloom's Taxonomy Level**	**Marks**
Q.01	a	Distinguish between (i) Periodic and Non-periodic signals (ii) Deterministic and Random signals.	L1	4 Marks
	b	Determine and sketch the even and odd components of the following signals: (i) $x[n] = \begin{cases} -2 + n^2, & -1 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$ (ii) $x(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2-t), & 1 < t < 2 \end{cases}$	L2	8 Marks
	c	Sketch and determine the energy of the following signals: (i) $x(t) = r(t+1) - r(t) - r(t-2) + r(t$ [illegible] $3)$ (ii) $x[n] = -u[n-1] + u[n-5]$	L3	8 Marks
		OR		
Q.02	a	Two signals x(t) and g(t) are shown in Fig.Q2(a). Express the signal x(t) in terms of g(t).  Fig. Q2(a)-i  Fig. Q2(a)-ii	L3	8 Marks
	b	Determine whether each of the following signals is periodic or not: If periodic, find the fundamental period: (i) $x[n] = \sin(3n)$ (ii) $x[n] = \cos\left(0.3\pi n + \frac{\pi}{4}\right)$ (iii) $x[n] = \sin\left(\frac{7\pi}{37}n\right)$	L2	6 Marks
	c	A signal $x(t) = (t+1)[u(t) - u(t-1)]$, is applied as input to a differentiator. Obtain the output signal y(t) and sketch the same.	L3	6 Marks
		Module-2		
Q. 03	a	Following signals represent input and impulse response of a continuous-time Linear and Time-Invariant (LTI): $x(t) = u(t) - u(t-3)$ $h(t) = e^{-[illegible]}[u(t+1) - u(t-1)]$ Obtain the output for the applied input.	L3	8 Marks

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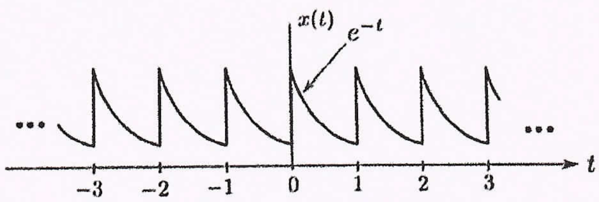
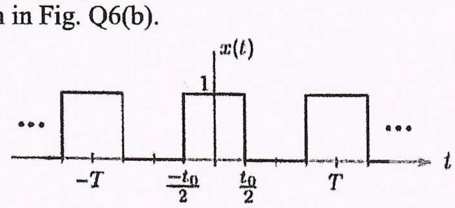
Scheme & Solution Prepared by Prof. Suraj Kadli

	b	Determine whether the following systems represented by input-output relations are Time-Invariant and Invertible: (i) $y[t] = x\left(\frac{t}{2}\right)$ (ii) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ (iii) $y[n] = x[n] + x[n-1]$ (iv) $y[n] = x[n]u[n]$	L2	8 Marks
	c	Perform Convolution operation on the following signals: $x[n] = \delta[n+1] - \delta[n-1] + \delta[n-3]$ $h[n] = \delta[n] - \delta[n-2]$ Sketch the resulting signal.	L3	4 Marks
		OR		
Q.04	a	Determine whether the following system represented by input-output relation is stable and causal: $y[n] = x[n+1] + x[n] + x[n-1]$.	L3	4 Marks
	b	Given $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n-1]$, obtain $y[n] = x[n] * h[n]$.	L3	8 Marks
	c	Show that distributive and associative laws hold good with respect to convolution operator in continuous-time domain.	L2	8 Marks
		Module-3		
Q. 05	a	Show that the step response of an LTI system is running integral of impulse response.	L2	4 Marks
	b	Determine whether the following systems represented by impulse response are causal and stable: (i) $h[n] = 5\,\delta[n]$ (ii) $h[n] = \left(\frac{1}{4}\right)^{\lvert n \rvert}$ (iii) $h[n] = \left(\frac{1}{2}\right)^{-n} u[-n]$	L2	6 Marks
	c	Find the complex Fourier coefficients X(k) for x(t) shown in Fig. Q5(c). Also sketch magnitude and phase spectra.  Fig. Q5(c)	L3	10 Marks
		OR		
Q. 06	a	Find complex Fourier series coefficients X(k) of the signal $x(t) = \lvert \sin \pi t \rvert$	L3	6 Marks
	b	Using the derivative property of continuous-time Fourier series, obtain X(k) of the signal x(t) shown in Fig. Q6(b).  Fig.Q6(b)	L3	8 Marks
	c	Obtain the step response for the following systems represented by impulse response: (i) $h[n] = \delta[n+3] - 2\,\delta[n] + 3\,\delta[n-2]$ (ii) $h[n] = u[n+2] - 2u[n] + u[n-3]$	L2	6 Marks

		Module-4		
Q. 07	a	Given $x(t) = \begin{cases} A, -T < t < T \\ 0, \text{ Otherwise} \end{cases}$ Obtain the Fourier transform of x(t). Also, sketch the magnitude and phase spectra.	L3	6 Marks
	b	Derive the Parseval relationship applicable to DTFT and mention its significance.	L2	6 Marks
	c	Given the Fourier transform of x(t), $X(j\omega) = \frac{j\omega}{-\omega^2+7j\omega+6}$, find the Fourier transform of the following signals: (i) $x(4t-8)$ (ii) $\int_{-\infty}^{t} x(\tau)d\tau$ (iii) $e^{-j100t}x(t)$	L3	8 Marks
		OR		
Q. 08	a	Show that DTFT is a periodic function of fundamental period 2π rad.	L2	4 Marks
	b	State and prove the following properties with respect to continuous-time Fourier Transform: (i) Frequency shifting (ii) Modulation	L2	8 Marks
	c	Find the DTFT of the following sequences: (i) $x[n] = n\,0.5^n u[n]$ (ii) $x[n] = \left(\frac{1}{4}\right)^n u[n-4]$ (ii) (iii) $x[n] = \left(\frac{1}{4}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n]$	L3	8 Marks
		Module-5		
Q. 09	a	Find the Z-transform of the signal $x[n] = (n(-0.5)^n u[n]) * 4^n u[-n]$.	L3	8 Marks
	b	Using long division method, find the inverse Z-transform of $X(z) = \frac{2+z^{-1}}{1-0.5z^{-1}}$ ROC: $\|z\| > 0.5$	L3	4 Marks
	c	An LTI system has impulse response $h[n] = 0.5^n u[n]$. Determine the input to the system if the output is given by $y[n] = 0.5^n u[n] + (-0.5)^n u[n]$	L3	8 Marks
		OR		
Q. 10	a	Determine the transfer function and a difference equation representation of an LTI system described by the impulse response: $h[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-2} u[n-1]$	L3	8 Marks
	b	A stable and causal LTI system is described by the difference equation: $y[n] + 0.25y[n-1] - 0.125\,y[n-2] = -2x[n] + 1.25\,x[n-1]$. Find the system impulse response.	L3	8 Marks
	c	Find the Z-transform of $x[n] = 0.5^n u[n] + 2^n u[-n-1]$.	L3	4 Marks

*Bloom's Taxonomy Level: Indicate as L1, L2, L3, L4, etc. It is also desirable to indicate the COs and POs to be attained by every bit of questions.

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Signals and Systems – 18EC45

Model Question Paper – 1 – Scheme and Solution.

Module – 1.

1a) Distinguish between, (i) Periodic and Non-periodic signals, (ii) Deterministic and Random signals. [Total – 4M]

→ i) Periodic and Non-periodic signals:-

A signal is said to be periodic if it repeats at regular interval of time. — 1M

$$x(t) = x(t+T), \quad \forall t$$ — 1M

$$x[n] = x[n+N], \quad \forall \text{ integer } n.$$

Eg:- sine wave

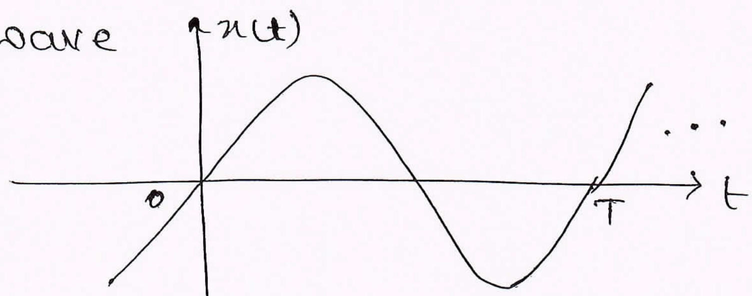


Fig 1(a)

T & N → Fundamental periods.

For discrete-time signals to be periodic, frequency should be rational multiple of 2π. — 1M

Signals which do not repeat at regular intervals are called as Non-periodic signals.

ii) Deterministic and Random signals:-

Deterministic signals are completely specified functions of time and, there is no uncertainity with respect to its value at any time.
Eg:- Sine wave, square pulse etc. — 1M

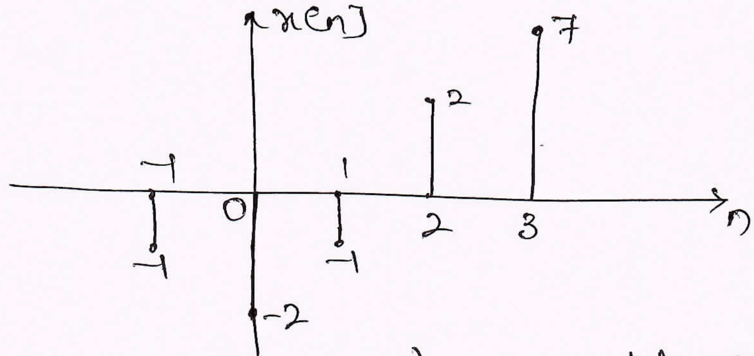
Random signals will be uncertain before they occur and they cannot be represent by any mathematical equation.
Eg:- Noise generated by electrical components.

1.b) Determine and sketch the even & odd components of the following signals:- 03

[Total - 8M].

i) $x[n] = \begin{cases} -2+n^2, & -1 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$, ii) $x(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2-t), & 1 < t < 2. \end{cases}$

→ i) $x[n] = \begin{cases} -2+n^2, & -1 \le n \le 3 \\ 0, & \text{otherwise.} \end{cases}$



— 1M

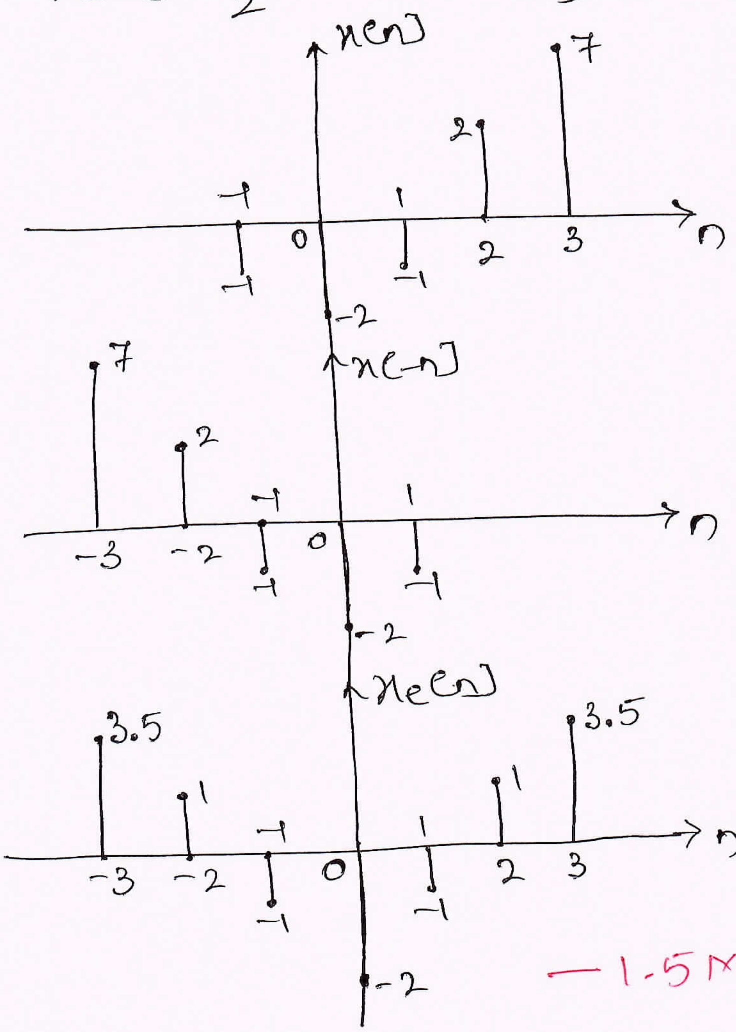
Fig. 1(b1)

Even component

$x_e[n] = \frac{1}{2}(x[n] + x[-n])$

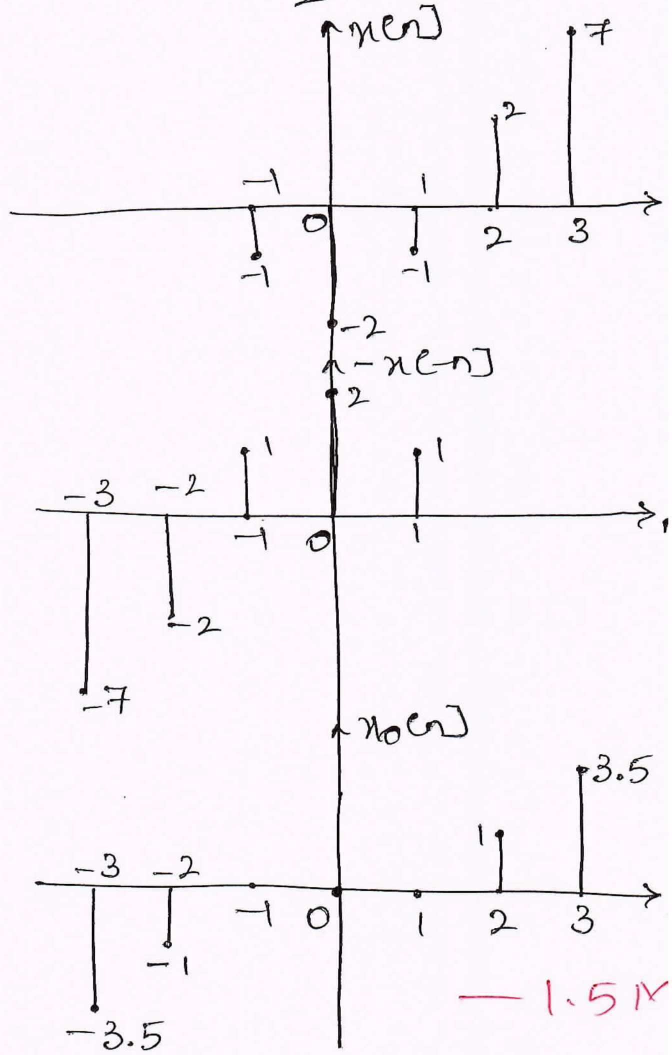
Odd component.

$x_o[n] = \frac{1}{2}(x[n] - x[-n])$



— 1.5M

Fig. 1(b2)



— 1.5M

Fig. 1(b3)

ii) $x(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2-t), & 1 < t < 2 \end{cases}$

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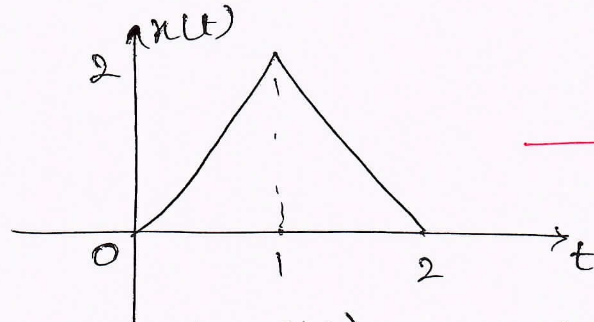


— 1M

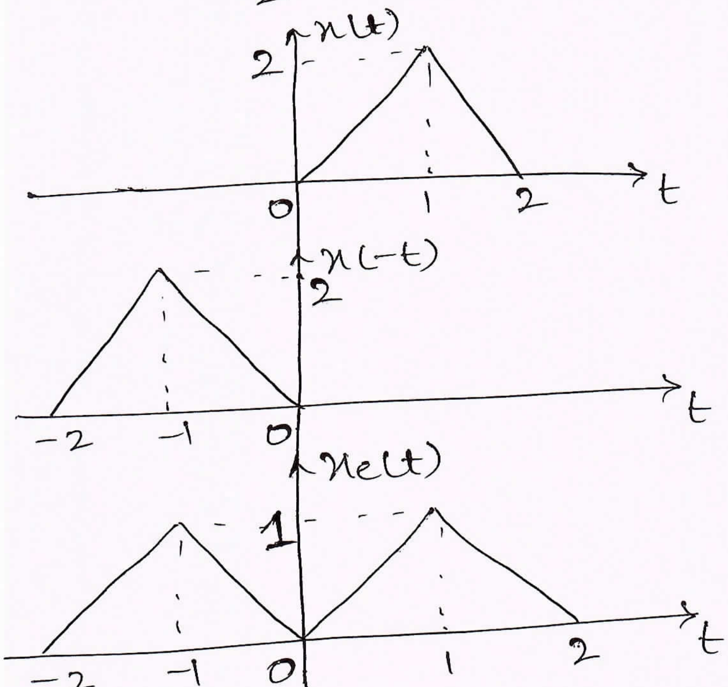
Fig 1(b4)

Even component

$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

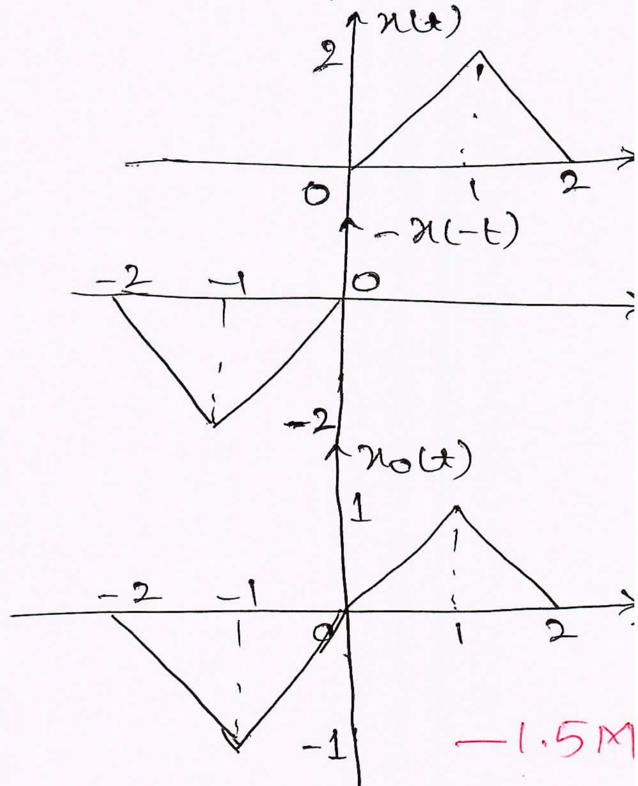
Odd component

$x_o(t) = \frac{1}{2}(x(t) - x(-t))$



— 1.5M

Fig 1(b5)



— 1.5M

Fig 1(b6)

1. c) Sketch and determine the energy of the following signals

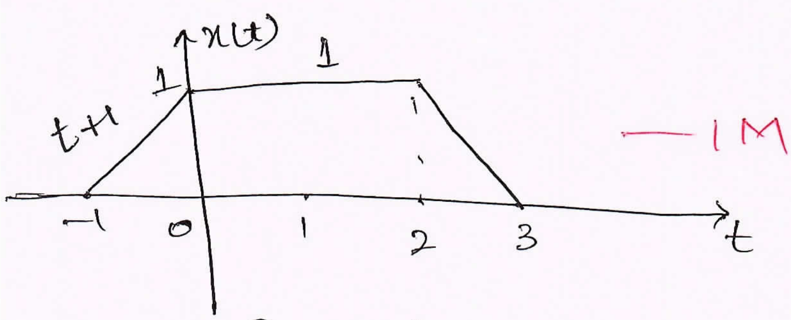
i) $x(t) = r(t+1) - r(t) - r(t-2) + r(t-3)$

ii) $x[n] = -u[n-1] + u[n-5]$.

[Total - 8M]

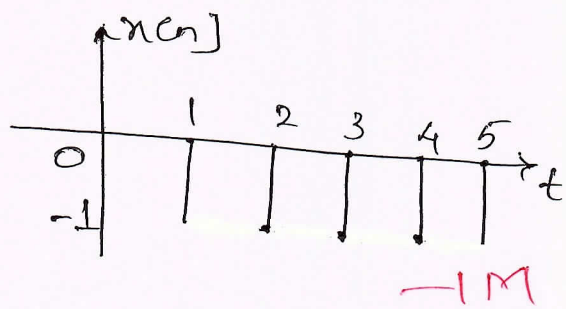
→ i) $x(t) = r(t+1) - r(t) - r(t-2) + r(t-3)$

ii) $x[n] = -u[n-1] + u[n-5]$



— 1M

Fig. 1(c1)



— 1M

Fig. 1(c2)

i) $E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$ — 1M

$E = \int_{-1}^{0} |t+1|^2 dt + \int_{0}^{2} 1 \, dt + \int_{2}^{3} |3-t|^2 dt.$ — 1M

$E = \left[\frac{t^3}{3} + t^2 + t\right]_{-1}^{0} + \left[t\right]_{0}^{2} + \left[\frac{t^3}{3} - 3t^2 + 9t\right]_{2}^{3}$ — 1M

$E = 0.33 + 2 + 0.34$

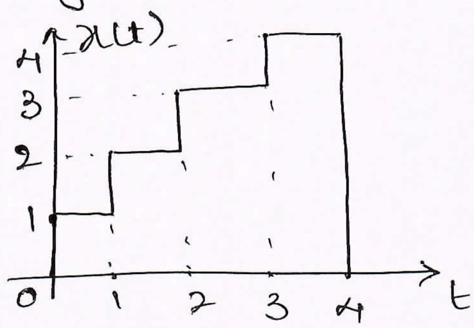
$E = 2.67$. — 1M

ii) $E = \sum_{n=-\infty}^{\infty} |x[n]|^2$ — 1M

$E = \sum_{n=1}^{5} |-1|^2 = \sum_{n=1}^{5} 1 = 5$ — 1M

OR

2.a) Two signals $x(t)$ and $g(t)$ are shown in Fig. 2(a). Express the signal $x(t)$ in terms of $g(t)$. [Total-8M]



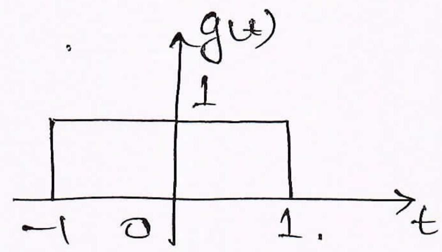
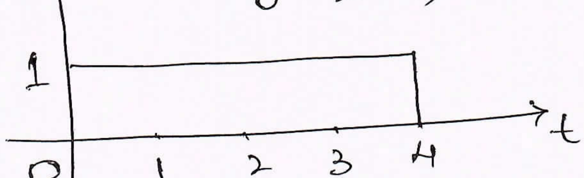


Fig. 2(a)

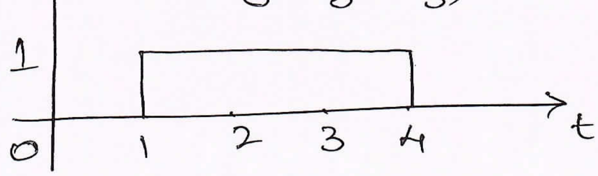
→ $x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t)$ — 1M

$x_1(t) = g\left(\frac{t}{2} - 1\right)$

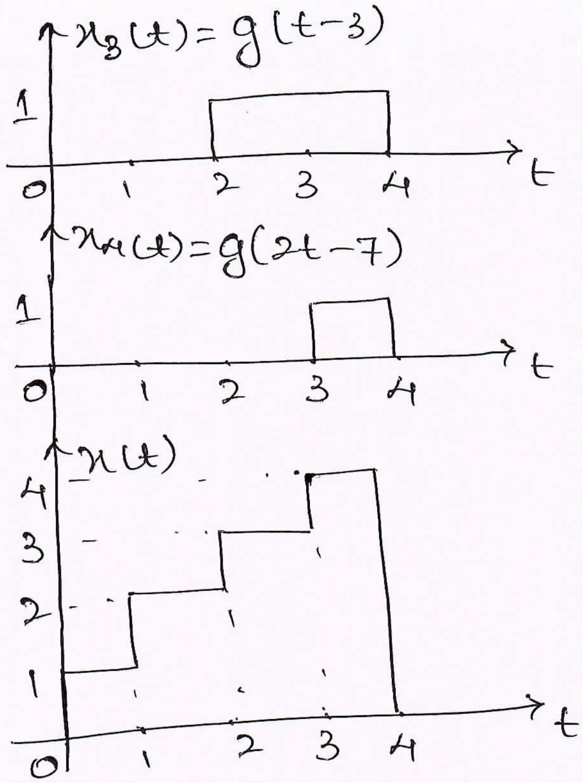


— 1M

$x_2(t) = g\left(\frac{2t}{3} - \frac{5}{3}\right)$



— 1M



— 1M

— 1M

— 1M

Fig 2(a1)

$$x(t) = g\left(\frac{t}{2} - 1\right) + g\left(\frac{2t}{3} - \frac{5}{3}\right) + g(t-3) + g(2t-7).$$ — 2M

-.b) Determine whether each of the following signals is periodic or not: If periodic, find the fundamental period:

i) $x[n] = \sin(3n)$, ii) $x[n] = \cos\left[0.3\pi n + \frac{\pi}{4}\right]$, iii) $x[n] = \sin\left(\frac{7\pi}{37}\right.$

[Total - 6M].

→ i) $x[n] = \sin(3n)$

$\Omega = 3$, which cannot be expressed as rational multiple of 2π. Therefore, $x[n] = \sin(3n)$ is Non-periodic.
— 2M

ii) $x[n] = \cos\left[0.3\pi n + \frac{\pi}{4}\right]$

$$\Omega = 0.3\pi = 2\pi\,\frac{0.15}{1} = 2\pi\,\frac{15}{100} = 2\pi\,\frac{m}{N}$$ — 1M

Frequency 0.3π can be expressed as rational multiple of 2π ∴ Signal is periodic with period, $N = 100$. — 1M

iii) $x[n] = \sin\left(\frac{7\pi}{37}n\right)$

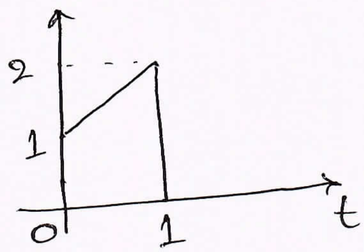
$$\Omega = \frac{7\pi}{37} = 2\pi\,\frac{3.5}{37} = 2\pi\,\frac{35}{370}$$
— 1M

Frequency is rational multiple of 2π, ∴ Signal is periodic with period, $N = 370$.
— 1M

2.c) A signal $x(t) = (t+1)[u(t) - u(t-1)]$, is applied as input to a differentiator. Obtain the output signal $y(t)$ and sketch the same. 11

[Total – 6M]

→ $x(t) = (t+1)[u(t) - u(t-1)]$.



— 1M

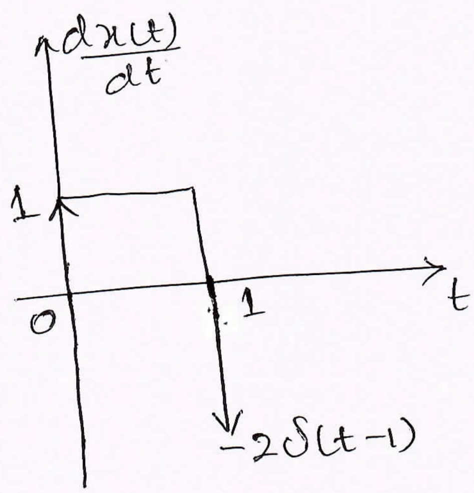
Fig. 2(c1)

$$\frac{dx(t)}{dt} = [t+1]\left[\delta(t) - \delta(t-1)\right] + [u(t) - u(t-1)] \times 1$$

$$= (t+1)\,\delta(t) - (t+1)\,\delta(t-1) + [u(t) - u(t-1)].$$

$$= (t+1)\Big|_{t=0}\delta(t) - (t+1)\Big|_{t=1}\delta(t-1) + [u(t) - u(t-1)].$$ — 2M

$$\frac{dx(t)}{dt} = \delta(t) - 2\,\delta(t-1) + [u(t) - u(t-1)].$$ — 1M



← 2M

Fig 2 (c2)

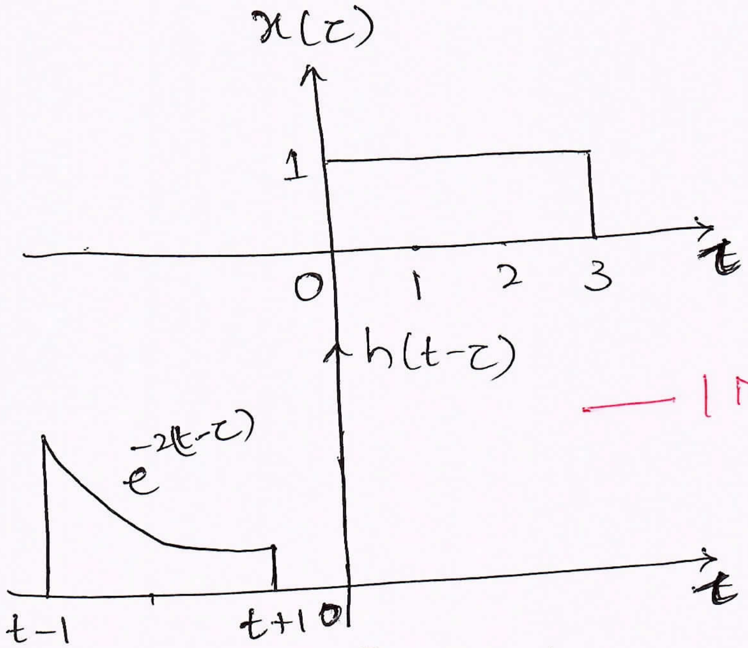
Module - 2

3. a) Following signals represent input and impulse response of a continuous-time Linear and Time-Invariant (LTI):

$$x(t) = u(t) - u(t-3) \quad , \quad h(t) = e^{-2t}[u(t+1) - u(t-1)]$$

Obtain the output for the applied input. [Total-8M]

→



— 1M

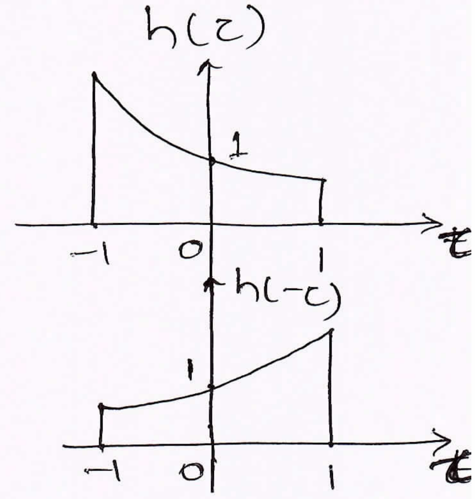


Fig 3(a1)

Fig 3(a)

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \quad , \quad \text{where, } w_t(\tau) = x(\tau) h(t-\tau)$$

a) for, $t+1 < 0$ or $t < -1$; the signals $x(\tau)$ & $h(t-\tau)$ do not overlap, $\therefore w_t(\tau) = 0$ — 1M

$\therefore y(t) = 0, \quad t < -1$

b) for, $t+1 > 0$ and $t+1 < 2$ i.e., $-1 < t < 1$

$$w_t(\tau) = e^{-2(t-\tau)} \quad , \quad 0 < \tau < t+1$$

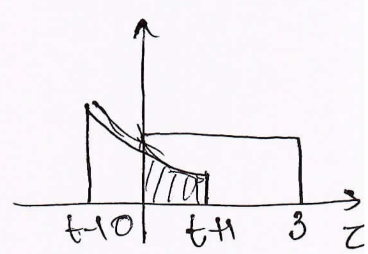


Fig 3(a2)

$$y(t) = \int_{-\infty}^{\infty} w_t(\tau)\, d\tau = \int_{0}^{t+1} e^{-2(t-\tau)}\, d\tau = e^{-2t} \int_{0}^{t+1} e^{2\tau}\, d\tau$$

$$y(t) = e^{-2t} \left[\frac{e^{2\tau}}{2} \right]_0^{t+1} = \frac{e^{-2t}}{2} [e^{2(t+1)} - 1] = \frac{e^2 - e^{-2t}}{2}$$

$$y(t) = \frac{7.38 - e^{-2t}}{2}, \quad -1 < t < 1$$ — 1M

> for, $1 < t < 2$ ($t+1 > 2$ and $t+1 < 3$).

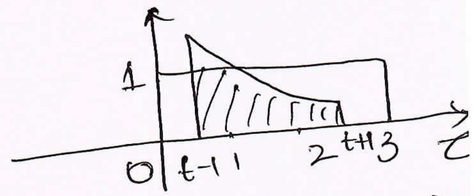


Fig 3(a3)

$$w_t(\tau) = e^{-2(t-\tau)}, \quad t-1 < \tau < t+1$$

$$y(t) = \int_{t-1}^{t+1} e^{-2(t-\tau)} d\tau = e^{-2t} \int_{t-1}^{t+1} e^{2\tau} d\tau = e^{-2t}\left[\frac{e^{2\tau}}{2}\right]_{t-1}^{t+1}$$

$$y(t) = \frac{e^{-2t}}{2}\left[e^{2t+2} - e^{2t-2}\right] = \frac{e^{2} - e^{-2}}{2} = 3.62, \quad 1 < t < 2$$ — 1M

iv) for, $t+1 > 3$ and $t-1 < 3$ i.e., $2 < t < 4$

$$w_t(\tau) = e^{-2(t-\tau)}, \quad t-1 < \tau < 3$$

$$y(t) = \int_{t-1}^{3} e^{-2(t-\tau)} d\tau = e^{-2t} \int_{t-1}^{3} e^{2\tau} d\tau = e^{-2t}\left[\frac{e^{2\tau}}{2}\right]_{t-1}^{3}$$

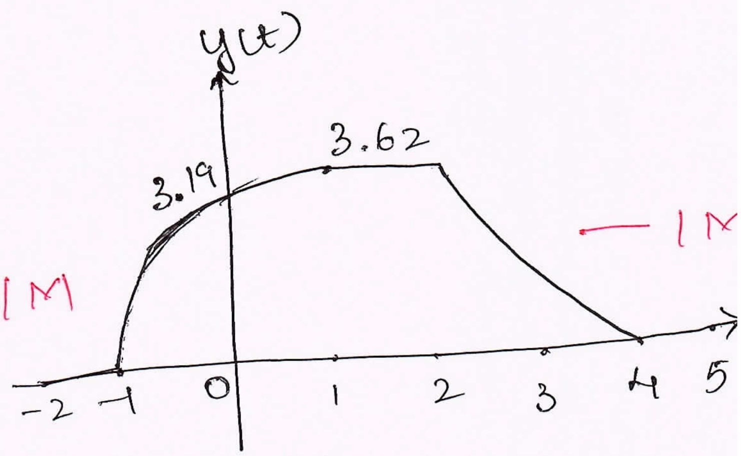
$$y(t) = \frac{e^{-2t}}{2}\left[e^{6} - e^{2t-2}\right] = \frac{e^{-2(t-3)} - 0.13}{2}, \quad 2 < t < 4$$ — 1M

e) for, $t-1 > 3$ or $t > 4$

$$w_t(\tau) = 0$$ — 1M

$$y(t) = 0, \quad t > 4$$

$$y(t) = \begin{cases} 0, & t < -1 \\ \dfrac{7.38 - e^{-2t}}{2}, & -1 < t < 1 \\ 3.62, & 1 < t < 2 \\ \dfrac{e^{-2(t-3)} - 0.13}{2}, & 2 < t < 4 \\ 0, & t > 4 \end{cases}$$ — 1M



— 1M

Fig 3(a4)

3.b) Determine whether the following systems represented by input-output relations are Time-invariant & Invertible: 1

i) $y(t) = x\left(\frac{t}{2}\right)$,   ii) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ [Total-8M]

iii) $y[n] = x[n] + x[n-1]$ iv) $y[n] = x[n]u[n]$.

→ i) $y(t) = x\left(\frac{t}{2}\right)$

* Time varying system, since the factor is divided by 2. — 2M
* It is invertible system.

ii) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

* It is time invariant system.

$y(t-t_0) = T\{x(t-t_0)\}$ — 2M

* differentiator is the inverse system

∴ It is invertible.

iii) $y[n] = x[n] + x[n-1]$

* $y[n-n_0] = T\{x[n-n_0]\}$ ∴ system is time invariant. — 2M
* It is an invertible system.

iv) $y[n] = x[n]u[n]$

* $T\{x[n-n_0]\} = x[n-n_0]u[n] \neq y[n-n_0]$ — 2M

∴ System is time variant.

* It is an identity system ∴ invertible.

3.c) Perform convolution operation on the following signals $x[n] = \delta[n+1] - \delta[n-1] + \delta[n-3]$, $h[n] = \delta[n] - \delta[n-2]$. Sketch the resulting signal. [Total-4M]

→ $x[n] = \delta[n+1] - \delta[n-1] + \delta[n-3]$

$h[n] = \delta[n] - \delta[n-2]$

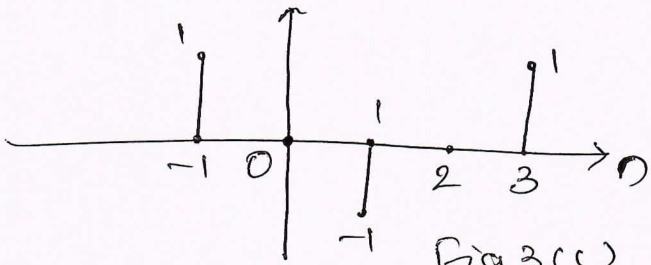


Fig 3(c)

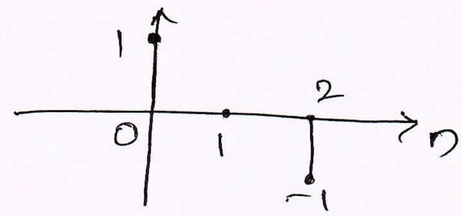


Fig 3(d)

— 1M

$x[n] = [1, \underset{\uparrow}{0}, -1, 0, 1]$

$h[n] = [\underset{\uparrow}{1}, 0, -1]$

	1	0	-1
1	1	0	-1
0	0	0	0
-1	-1	0	1
0	0	0	0
1	1	0	-1

— 1M

$y[n] = x[n] * h[n] = [1, \underset{\uparrow}{0}, -2, 0, 2, 0, -1]$ — 1M

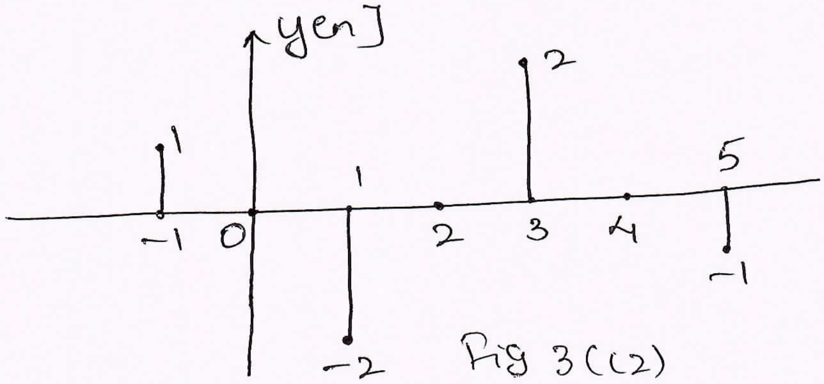


Fig 3(c2)

— 1M

'OR'

4.a) Determine whether the following system represented by input-output relation is stable and causal:
$y[n] = x[n+1] + x[n] + x[n-1]$. [Total - 4M]

→ $y[n] = x[n+1] + x[n] + x[n-1]$

* Assuming $|x[n]| \le M_x < \infty$

$$|y[n]| = |x[n+1] + x[n] + x[n-1]|$$

$$|y[n]| \le |x[n+1]| + |x[n]| + |x[n-1]| \quad \text{— 1M}$$

$$|y[n]| \le [M_x + M_x + M_x] = 3M_x.$$

$\therefore$ $|y[n]| \le 3M_x < \infty$ — 1M

$\therefore$ System is stable system. — 1M

* Output $y[n]$ depends on future value of the input $x[n+1]$ $\therefore$ System is non-causal. — 1M

4. b) Given, $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n-1]$, obtain $y[n] = x[n] * h[n]$.

[Total-8M]

→ $$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \quad \text{— 1M}$$

$$= \sum_{k=-\infty}^{\infty} \alpha^n u[n].\, \beta^{n-k} u[n-k-1] \quad \text{— 1M}$$

$$y[n] = \beta^n \sum_{k=-\infty}^{\infty} (\alpha\beta^{-1})^k [u[n]\, u[n-k-1]] \quad \text{— 1M}$$

$$u[n]\, u[n-k-1] = \begin{cases} 1, & n \ge k \text{ \& } n \ge 1 \\ 0, & n < k \text{ \& } n < 1 \end{cases} \quad \text{— 1M}$$

$$y[n] = \beta^n \sum_{k=1}^{n} (\alpha\beta^{-1})^k = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for } \alpha \ne \beta$$

— 2M

and, $$y[n] = \beta^n \sum_{k=1}^{n} \alpha^k \beta^{-k} = \beta^n \sum_{k=1}^{n} 1 = \alpha^n [n+1] u[n], \text{ for } \alpha = \beta$$

— 2M

4.c) Show that distributive and associative laws hold good with respect to convolution operator in continuous-time domain. 27

[Total - 8M].

→ *Distributive property of convolution integral:-

$$x(t) * [h_1(t) \cancel{*} h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).$$ — 1M

$$x(t) * [h_1(t) + h_2(t)] = \int_{-\infty}^{\infty} x(\tau)[h_1(t-\tau) + h_2(t-\tau)] d\tau$$ — 1M

$$= \int_{-\infty}^{\infty} x(\tau) h_1(t-\tau) d\tau + \int_{-\infty}^{\infty} x(\tau) h_2(t-\tau) d\tau$$ — 1M

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t).$$ — 1M

therefore distributive property hold good for convolution integral.

* Associative property of convolution integral:-

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)].$$ — 1M

let $f_1(t) = x(t) * h_1(t)$

$f_2(t) = h_1(t) * h_2(t)$

$$f_1(t) = \int_{-\infty}^{\infty} x(\tau) h_1(t-\tau) d\tau$$ — 1M

and, $[x(t) * h_1(t)] * h_2(t) = f_1(t) * h_2(t)$

$$= \int_{\sigma=-\infty}^{\infty} f_1(\sigma) h_2(t-\sigma) d\sigma$$

$$= \int_{\sigma=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} x(\tau) h_1(t-\tau) d\tau \right] h_2(t-\sigma) d\sigma$$ — 1M

let us use $\lambda = \sigma - \tau$. or $\sigma = \tau + \lambda$.

$$[x(t) * h_1(t)] * h_2(t) = \int_{\tau=-\infty}^{\infty} x(\tau) \left[\int_{\lambda=-\infty}^{\infty} h_1(\lambda) h_2(t-\tau-\lambda) d\lambda \right] d\tau$$ — 1M

$$f_2(t) = h_1(t) * h_2(t)$$

$$= \int_{\lambda=-\infty}^{\infty} h_1(\lambda)\, h_2(t-\lambda)\, d\lambda$$

$$f_2(t-\tau) = \int_{\lambda=-\infty}^{\infty} h_1(\lambda)\, h_2(t-\tau-\lambda)\, d\lambda.$$

$$\therefore [x(t) * h_1(t)] * h_2(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, f_2(t-\tau)\, d\tau. \quad \text{— 1M}$$

$$= x(t) * f_2(t)$$

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)].$$

Associative law hold good for convolution integral.

Module - 3

5.a) Show that the step response of an LTI system is running integral of impulse response. [Total - 4M

→ Step response is defined as the output due to a unit-step input signal. Let h[n] be the impulse response of a discrete-time LTI system, and step response be denoted as s[n], then, — 1M

$$u[n] \longrightarrow \boxed{h[n]} \longrightarrow s[n]$$

$$s[n] = u[n] * h[n] \quad \text{— 1M}$$

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]\, u[n-k]$$

$$u[n-k] = \begin{cases} 1, & n-k \geq 0 \text{ or } k \leq n \\ 0, & k > n \end{cases}$$

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

Therefore, ~~step~~ response is running sum of impulse response

Similarly,

$$s(t) = u(t) * h(t)$$

$$s(t) = h(t) * u(t)$$

$$s(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau \qquad \text{— 1M}$$

$$u(t-\tau) = \begin{cases} 1, & t-\tau \geq 0 \text{ or } \tau \leq t \\ 0, & \tau > t \end{cases}$$

$$\therefore s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau. \qquad \text{— 1M}$$

$\therefore$ step response for continuous linear time invariant system is running integral of impulse response.

5.b) Determine whether the following systems represented by impulse response are causal and stable:
i) $h[n] = 5\delta[n]$, ii) $h[n] = \left(\frac{1}{4}\right)^{|n|}$ & iii) $h[n] = \left(\frac{1}{2}\right)^{-n} u[-n]$
[Total - 6M]

→ i) $h[n] = 5\delta[n]$.

* System is causal, since $h[n]$ is zero for $n<0$. — 1M
* Let $S = \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |5\delta[n]| = 5 < \infty$ — 1M

$\therefore$ system is stable.

ii) $h[n] = \left(\frac{1}{4}\right)^{|n|}$

* System is non-causal, because $h[n] \neq 0$, for $n<0$ — 1M
* $S = \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} \left|\frac{1}{4}^{|n|}\right| = \sum_{n=-\infty}^{-1} 0.25^{-n} + \sum_{n=0}^{\infty} 0.25^{n}$

$n=-m$ in first summation $\qquad S = \sum_{m=1}^{\infty} (0.25)^{m} + \sum_{n=0}^{\infty} (0.25)^{m}$

$$S = \frac{0.25}{1-0.25} + \frac{1}{1-0.25}$$

$$S = 1.66 < \infty \qquad \text{— 1M}$$

$\therefore$ system is stable.

iii) $h[n] = \left(\frac{1}{2}\right)^{-n} u[-n]$

* System is noncausal because $h[n] \neq 0$ for $n<0$ — 1M

* $S = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{-k} u[-k] = \sum_{k=-\infty}^{0} \left(\frac{1}{2}\right)^{-k} = \sum_{k=0}^{\infty} \frac{1}{2} = \frac{1}{1-\frac{1}{2}} = 2 < \infty$ — 1M

$\therefore$ system is stable.

5.c) Find the complex fourier coefficients $X[k]$ for $x(t)$ shown in fig. 5(c). Also sketch magnitude and phase spectra.

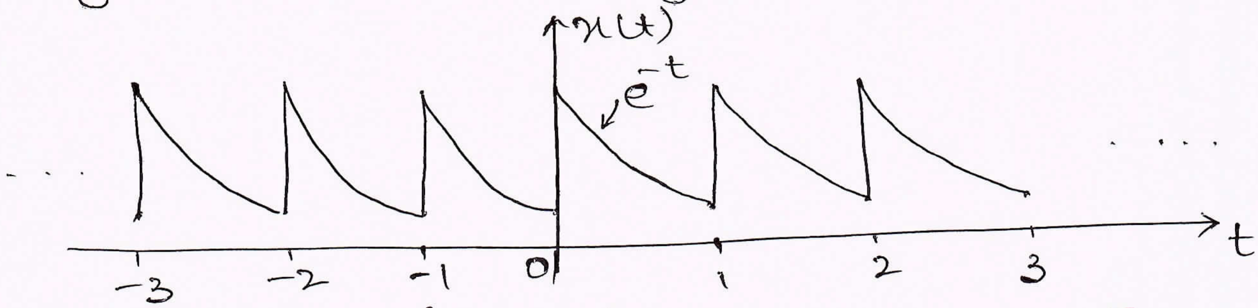


fig. 5(c)

[Total—10M]

→ Fundamental period, $T = 1$ $\therefore \omega_0 = \frac{2\pi}{T} = 2\pi$ — 1M

$x(t) = e^{-t}$ for $0 \le t \le 1$

$$X[k] = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt. \qquad \text{— 1M}$$

$$X[k] = \int_0^1 e^{-t} e^{-jk\omega_0 t}\, dt$$

$$X[k] = \int_0^1 e^{-(1+j2\pi k)t}\, dt = \left[\frac{e^{-(1+j2\pi k)t}}{-(1+j2\pi k)}\right]_0^1$$

— 1M

$$X[k] = \frac{e^{-(1+j2\pi k)} - 1}{-(1+j2\pi k)} = \frac{1}{1+j2\pi k}\left[1 - e^{-(1+j2\pi k)}\right] \quad \text{— 1M} \qquad 35$$

$$X[k] = \frac{1}{1+j2\pi k}\left[1 - e^{-1} e^{-j2\pi k}\right], \quad e^{-j2\pi k} = 1.$$

$$X[k] = \frac{1-e^{-1}}{1+j2\pi k} = \frac{0.632}{1+j2\pi k}.$$

Fourier series coefficients are given by

$$\boxed{X[k] = \frac{0.632}{1+j2\pi k}} \quad \text{— 1M}$$

$X[0] = 0.632$, $|X[0]| = 0.632$, $\angle X[0] = 0$

$X[1] = 0.015 - 0.098j$, $|X[1]| = 0.099$, $\angle X[1] = -1.41$

$X[-1] = 0.015 + 0.098j$, $|X[-1]| = 0.099$, $\angle X[-1] = 1.41$

$X[2] = 0.003 - 0.049j$, $|X[2]| = 0.049$, $\angle X[2] = -1.5$

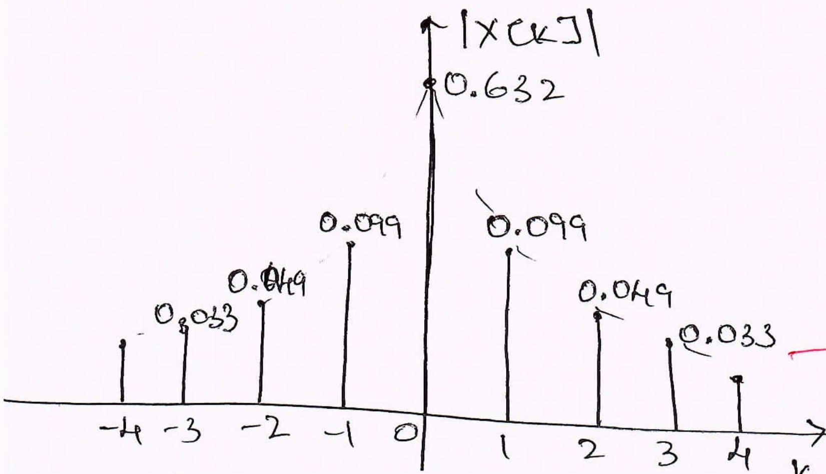
$X[-2] = 0.003 + 0.049j$, $|X[-2]| = 0.049$, $\angle X[-2] = 1.5$ ← 3M

$X[3] = 0.001 - 0.033j$, $|X[3]| = 0.033$, $\angle X[3] = -1.54$

$X[-3] = 0.001 + 0.033j$, $|X[-3]| = 0.033$, $\angle X[-3] = 1.54$

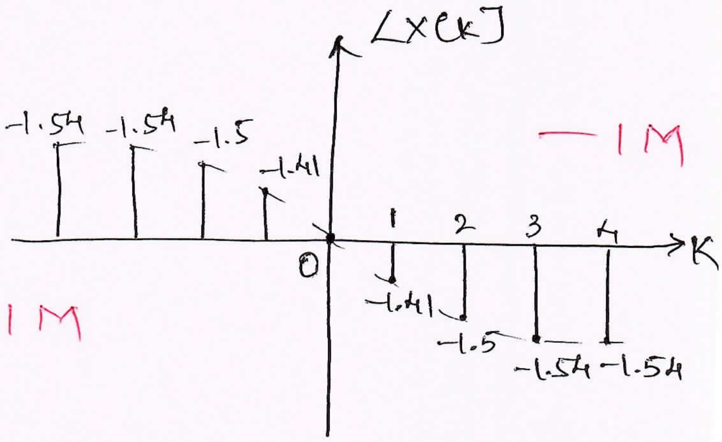
$X[4] = 0.0009 - 0.025j$, $|X[4]| = 0.025$, $\angle X[4] = -1.54$

$X[-4] = 0.0009 + 0.025j$, $|X[-4]| = 0.025$, $\angle X[-4] = 1.54.$



— 1M

Magnitude spectrum Fig 5(c1)



— 1M

Phase spectrum.

'OR'

01

6.a) Find complex fourier series coefficients $X[k]$ of the signal $x(t) = |\sin \pi t|$ [Total - 6M]

$\rightarrow$ $x(t) = |\sin \pi t|$

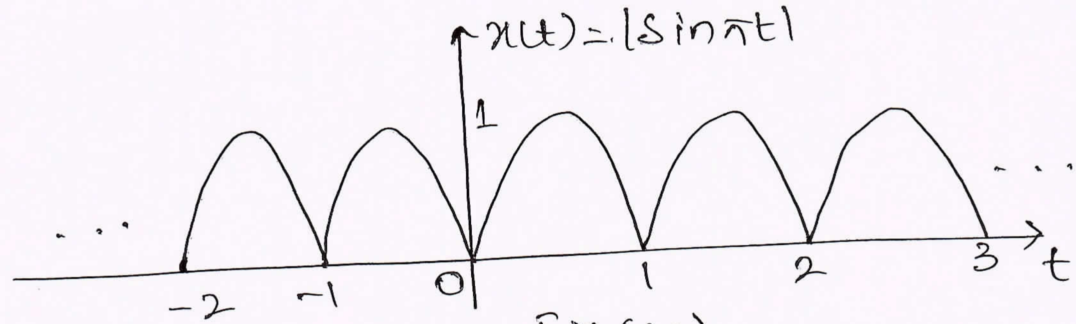


Fig 6(a)

— 1 M

$T = 1, \quad \omega_0 = \frac{2\pi}{T} = 2\pi$ — 1 M

$$X[k] = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$X[k] = \int_0^1 |\sin \pi t|\, e^{-jk2\pi t}\, dt = \int_0^1 \left[\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right] e^{-jk2\pi t}\, dt.$$ — 1 M

$$X[k] = \frac{1}{2j}\int_0^1 \left[e^{j\pi(1-2k)t} - e^{-j\pi(1+2k)t}\right] dt$$

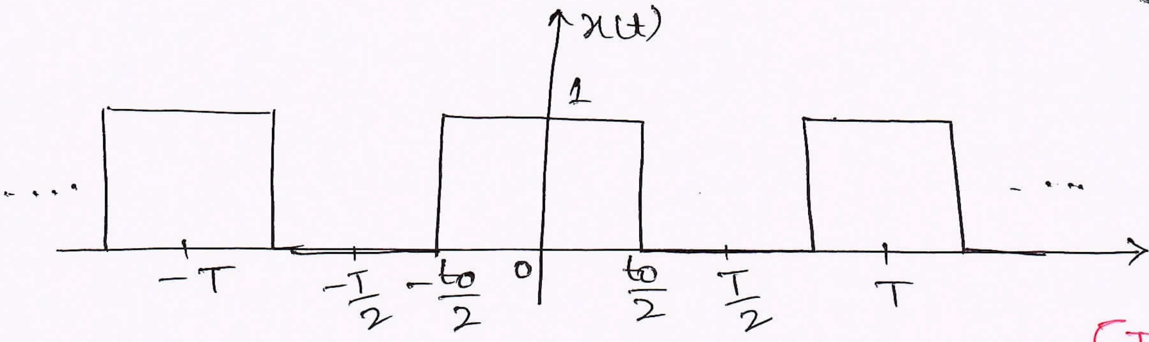
$$X[k] = \frac{1}{2j}\left[\left.\frac{e^{j\pi(1-2k)t}}{j\pi(1-2k)}\right|_0^1 + \left.\frac{e^{-j\pi(1+2k)t}}{j\pi(1+2k)}\right|_0^1\right]$$ — 1 M

$$X[k] = \frac{1}{2j}\cdot\frac{1}{j}\left[\frac{e^{j\pi(1-2k)} - 1}{\pi(1-2k)} + \frac{e^{-j\pi(1+2k)} - 1}{\pi(1+2k)}\right]$$ — 1 M

$$\boxed{X[k] = \frac{0.5 - 0.5\, e^{j\pi(1-2k)}}{\pi(1-2k)} + \frac{0.5 - 0.5\, e^{-j\pi(1+2k)}}{\pi(1+2k)}.}$$

— 1M

6.b) Using the derivative property of continuous-time Fourier series, obtain X[k] of the signal x(t) shown in Fig. Q6(b) 03



[Total-8M]

Fig. 6(b)

→ Differentiating x(t) wrt time we get the signal y(t) as shown in figure a.

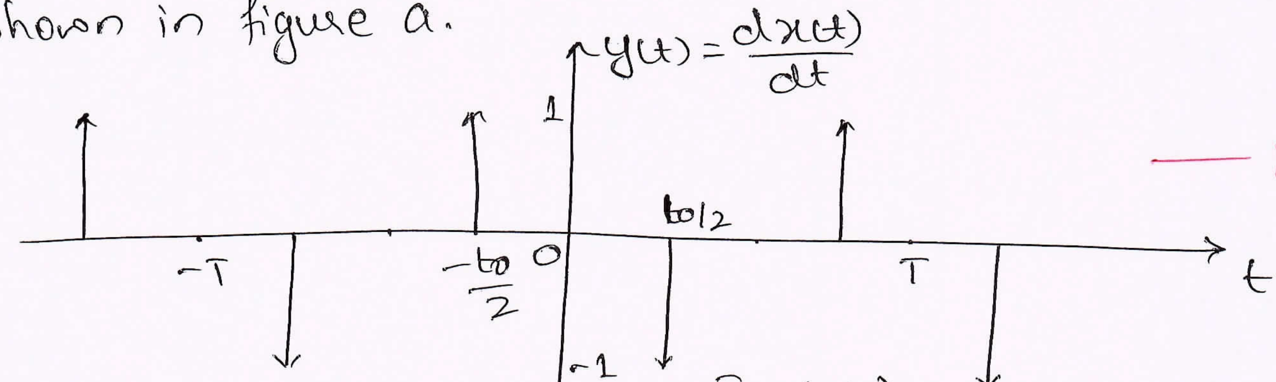


Fig 6(bi)

— 1M

$$\therefore \; y(t) = \delta\left(t+\frac{t_0}{2}\right) - \delta\left(t-\frac{t_0}{2}\right), \quad -\frac{T}{2} < t < \frac{T}{2}$$ — 1M

$$\therefore \; Y[k] = \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} y(t)\, e^{jk\omega_0 t}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \left[\delta\left(t+\frac{t_0}{2}\right) - \delta\left(t-\frac{t_0}{2}\right)\right] e^{jk\omega_0 t}\, dt.$$ — 1M

$$= \frac{1}{T}\left(\int_{-T/2}^{T/2} \delta\left(t+\frac{t_0}{2}\right) e^{jk\omega_0 t}\, dt - \int_{-T/2}^{T/2} \delta\left(t-\frac{t_0}{2}\right) e^{-jk\omega_0 t}\, dt\right)$$ — 1M

Applying sifting property of impulse.

$$Y[k] = \frac{1}{T}\left[e^{jk\omega_0 \frac{t_0}{2}} - e^{-jk\omega_0 \frac{t_0}{2}}\right] = \frac{2j}{T}\sin\left(k\omega_0 \frac{t_0}{2}\right)$$

— 1M

Using time-differentiation property,

$$Y[k] = jk\omega_0 X[k].$$

$$X[k] = \frac{1}{jk\omega_0} \cdot Y[k] \quad \text{— 1M}$$

$$X[k] = \frac{1}{jk\omega_0} \cdot \frac{2j}{T} \sin\left(k\omega_0 \frac{t_0}{2}\right) = \frac{2}{k\omega_0 T} \sin\left(k\omega_0 \frac{t_0}{2}\right)$$

— 1M

$$X[k] = \frac{2}{k 2\pi f_0 T} \cdot \sin\left(k 2\pi f_0 \frac{t_0}{2}\right) = \frac{t_0}{T} \frac{\sin(\pi k f_0 t_0)}{\pi k f_0 t_0}.$$

$$\boxed{X[k] = \frac{t_0}{T} \operatorname{sinc}(k f_0 t_0)} \quad \text{— 1M}$$

6.c) Obtain the step response for the following systems represented by impulse response:

i) $h[n] = \delta[n+3] - 2\delta[n] + 3\delta[n-2]$

ii) $h[n] = u[n+2] - 2u[n] + u[n-3]$. [Total – 6M]

→ i) $h[n] = \delta[n+3] - 2\delta[n] + 3\delta[n-2]$.

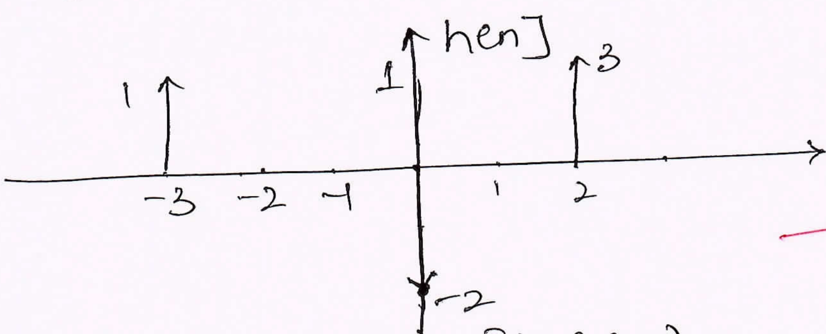


Fig 6(c)

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

— 1M

for, $n < -3$, $s[n] = 0$

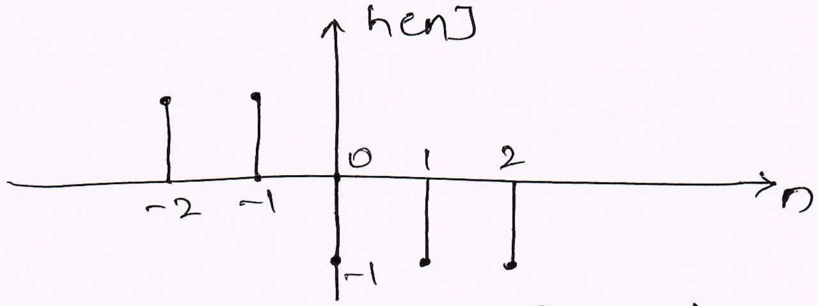
$n = -3$, $s[n] = 1$

$n = 0$, $s[n] = 1 - 2 = -1$ — 1M

$n \geq 2$, $s[n] = 1 - 2 + 3 = 2$

$$s[n] = \begin{cases} 0, & n < -3 \\ 1, & n = -3 \\ -1, & n = 0 \\ 2, & n \geq 2 \end{cases} \quad \text{— 1M}$$

ii) $h[n] = u[n+2] - 2u[n] + u[n-3]$



— 1M

Fig 6(c(i))

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

for, $n < -2$, $s[n] = 0$

$n = -2$, $s[n] = 1$

$n = -1$, $s[n] = 1 + 1 = 2$

$n = 0$, $s[n] = 1 + 1 - 1 = 1$ — 1M

$n = 1$, $s[n] = 1 + 1 - 1 - 1 = 0$

$n \geq 2$, $s[n] = 1 + 1 - 1 - 1 - 1 = -1$

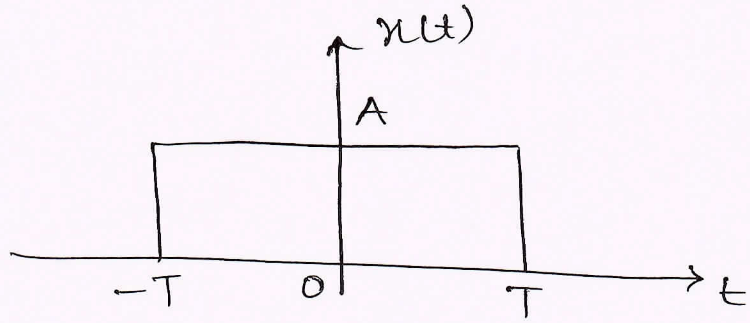
$$s[n] = \begin{cases} 0, & n < -2 \\ 1, & n = -2 \\ 2, & n = -1 \\ 1, & n = 0 \\ 0, & n = 1 \\ -1, & n \geq 2 \end{cases}$$ — 1M

Module - 4

7. a) Given, $x(t) = \begin{cases} A, & -T < t < T \\ 0, & \text{otherwise} \end{cases}$

Obtain the Fourier Transform of $x(t)$, Also, sketch the magnitude and phase. [Total – 6M]

→



— 1M

Fig 7(a)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-T}^{T} A\, e^{-j\omega t}\, dt = A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T}^{T} \quad \text{— 1M}$$

$$X(j\omega) = \frac{A}{-j\omega}\left[e^{-j\omega T} - e^{j\omega T}\right] = \frac{2A}{\omega}\,\frac{e^{j\omega T} - e^{-j\omega T}}{2j}$$

$$X(j\omega) = \frac{2A}{\omega}\sin(\omega T) \quad \text{— 1M}$$

rearranging to obtain sinc function.

$$X(j\omega) = 2AT\,\frac{\sin\left(\pi\frac{\omega T}{\pi}\right)}{\pi\frac{\omega T}{\pi}}$$

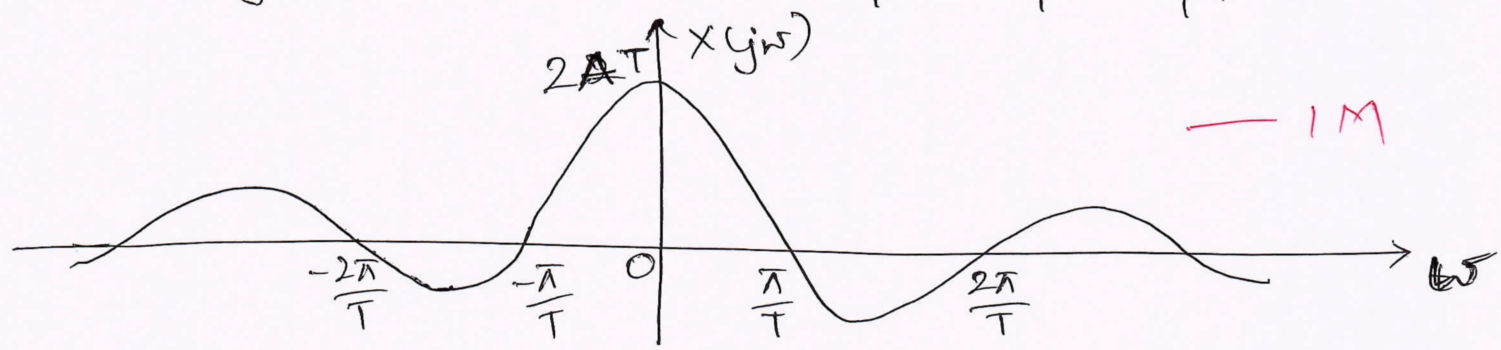
$$X(0) = \int_{-T}^{T} A \times e^{0}\, dt$$

$$\boxed{X(j\omega) = 2\cancel{A}T\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right)} \quad \text{— 1M}$$

$$X(0) = 2\cancel{A}T.$$

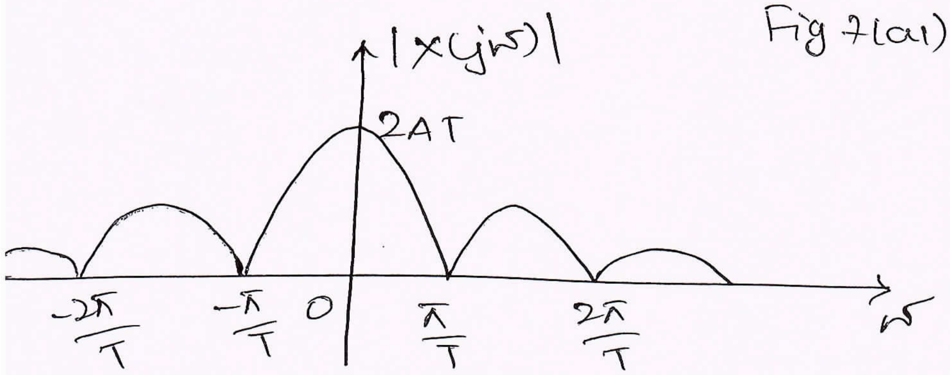
$$|X(j\omega)| = 2AT\,\mathrm{sinc}\left(\frac{\omega T}{\pi}\right) \quad \& \quad \angle X(j\omega) = 0.$$

function goes to zero at $\omega = \pm\frac{\pi}{T}, \pm\frac{2\pi}{T}, \pm\frac{3\pi}{T}, \ldots$



— 1M

Fig 7(a1)



— 1M

Fig 7(a2)

7.b) Derive the Parseval relationship applicable to DTFT and mention its significance. [Total - 6M]

→ Parseval's relationship of DTFT :-

If, $x[n] \xleftrightarrow{DTFT} X(\Omega)$

then, $E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |X(\Omega)|^2 d\Omega.$ — 1M

$|X(\Omega)|^2$ is known as energy density spectrum of the signal $x[n]$ & E is the total energy content of the sequence $x[n]$.

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, x^*[n]. \qquad — (1) \quad — 1M$$

w.k.t, $x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\Omega)\, e^{j\Omega n}\, d\Omega.$

taking conjugate on both sides

$$x^*[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X^*(\Omega)\, e^{-j\Omega n}\, d\Omega. \qquad — (2) \quad — 1M$$

Using (2) in (1)

$$E = \sum_{n=-\infty}^{\infty} x[n]\, \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X^*(\Omega)\, e^{-j\Omega n}\, d\Omega \qquad — 1M$$

Interchanging order of summation & integration

$$E = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X^*(\Omega) \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}\, d\Omega.$$

$$E = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X^*(\Omega) X(\Omega) \, d\Omega.$$

$$\therefore E = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |X(\Omega)|^2 \, d\Omega. \quad \text{— 1M}$$

Significance:- Parseval's theorem tells us that the DTFT is a linear transform that preserves the norm of a signal. — 1M

c) Given the Fourier transform, of $x(t)$, $X(j\omega) = \dfrac{j\omega}{-\omega^2 + 7j\omega + 6}$,

find the Fourier transform of the following signals:
i) $x(4t-8)$, ii) $\int_{-\infty}^{t} x(\tau)\, d\tau$, & iii) $e^{j100t} x(t)$.

[Total-8M]

→ i) $x(4t-8)$

using time-shift property,

$$y(t) = x(t-8) \xleftrightarrow{FT} Y(j\omega) = e^{j\omega 8} X(j\omega) \quad \text{— 1M}$$

$$Y(j\omega) = e^{j\omega 8} \frac{j\omega}{-\omega^2 + 7j\omega + 6} = \frac{e^{j\omega 8} \, j\omega}{-\omega^2 + 7j\omega + 6} \quad \text{— 1M}$$

using time-scaling property,

$$z(t) = y(4t) = x(4t-8) \xleftrightarrow{FT} Z(j\omega) = \frac{1}{|4|} Y\left(j\frac{\omega}{4}\right)$$

$$Z(j\omega) = \frac{1}{4} \frac{e^{-j2\omega} \, j\frac{\omega}{4}}{-\frac{\omega^2}{16} + \frac{7j\omega}{4} + 6} \quad \text{— 1M}$$

$$\boxed{Z(j\omega) = \frac{e^{j2\omega} \, j\omega}{-\omega^2 + 28j\omega + 96}} \quad \text{— 1M}$$

ii) $\int_{-\infty}^{t} x(\tau)d\tau$

using integration property,

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(j\omega)$$ — 1M

$$\therefore Y(j\omega) = \frac{1}{\cancel{j\omega}} \frac{\cancel{j\omega}}{-\omega^2+7j\omega+6} + \pi \cancel{X(0)}^{0} \delta(j\omega)$$

$$\therefore \boxed{Y(j\omega) = \frac{1}{-\omega^2+7j\omega+6}}$$ — 1M

iii) $e^{-j100t} x(t)$

using frequency-shift property,

$$y(t) = e^{-j100t} x(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega+100) = X(j\omega+100)$$

$$Y(j\omega) = \frac{j(\omega+100)}{-(\omega+100)^2 + 7j(\omega+100) + 6}$$ — 1M

$$\boxed{Y(j\omega) = \frac{j(\omega+100)}{-\omega^2+(7j-200)\omega+700j-9994}}$$ — 1M

'OR'

8.a) Show that DTFT is a periodic function of fundamental perio[d] 2π rad. [Total - 4 M].

→ DTFT, $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$ — 1 M

Change, Ω to $\Omega + 2\pi k$

$$X(\Omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega + 2\pi k)n}$$

$$X(\Omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} e^{-j2\pi k n}$$ — 1 M

$e^{-j2\pi kn} = \cos(\underbrace{2\pi kn}_{1}) - j\sin(\underbrace{2\pi kn}_{0})$

k & n are integers

$\therefore e^{-j2\pi kn} = 1.$

$$X(\Omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$ — 1 M

$$\therefore \boxed{X(\Omega + 2\pi k) = X(\Omega)}$$ — 1 M

Discrete time Fourier Transform is periodic with a period equal to 2π.

.b) State and prove the following properties with respect to continuous-time Fourier Transform: i) Frequency shifting. & ii) Modulation. [Total-8M]

→ i) Frequency shifting property of CTFT:-

If, $x(t) \xleftrightarrow{FT} X(j\omega)$ — 1M

then, $y(t) = e^{j\beta t} x(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega - \beta)$

→ CTFT, $Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$

$= \int_{-\infty}^{\infty} e^{j\beta t} x(t) e^{-j\omega t} dt$ — 1M

$= \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\beta)t} dt.$ — 1M

$\therefore Y(j\omega) = X(j\omega - \beta)$ — 1M

Hence proved.

ii) Modulation property of DTFT:-

If, $x(t) \xleftrightarrow{FT} X(j\omega)$

and, $y(t) \xleftrightarrow{FT} Y(j\omega)$ — 1M

then, $z(t) = x(t)y(t) \xleftrightarrow{FT} Z(j\omega) = \frac{1}{2\pi} [X(j\omega) * Y(j\omega)].$

→ CTFT, $Z(j\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$

$= \int_{-\infty}^{\infty} x(t) y(t) e^{-j\omega t} dt$ — 1M

From the definition of inverse Fourier Transform we have,

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\lambda) e^{j\lambda t} d\lambda$ — 1M

$$Z(j\omega) = \int_{t=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X(j\lambda) e^{j\lambda t} d\lambda \right] y(t) e^{-j\omega t} dt.$$

interchanging the order of integration

$$Z(j\omega) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X(j\lambda) \int_{t=-\infty}^{\infty} y(t) e^{-j(\omega-\lambda)t} dt\, d\lambda.$$

$$Z(j\omega) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} X(j\lambda) Y(j\omega-\lambda) d\lambda.$$

$$\therefore \boxed{Z(j\omega) = \frac{1}{2\pi} [X(j\omega) * Y(j\omega)]} \quad -1M$$

because, $Y(j\omega-\lambda) = \int_{t=-\infty}^{\infty} y(t) e^{-j(\omega-\lambda)t} dt$

Multiplication in time-domain is equivalent to convolution in frequency domain.

8. c) Find the DTFT of the following sequences:-

i) $x[n] = n\, 0.5^n u[n]$, ii) $x[n] = \left(\frac{1}{4}\right)^n u[n-4]$

iii) $x[n] = \left(\frac{1}{4}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n]$. [Total-8M]

→ i) $x[n] = n\, 0.5^n u[n]$.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} n \left(\frac{1}{2}\right)^n e^{-j\Omega n} = \sum_{n=0}^{\infty} n \left(\frac{1}{2} e^{-j\Omega}\right)^n \quad -1M$$

$$\therefore \boxed{X(\Omega) = \frac{\frac{1}{2} e^{-j\Omega}}{\left(1 - \frac{1}{2} e^{-j\Omega}\right)^2}} \quad -1M$$

$\because \sum_{n=0}^{\infty} n\beta^n = \frac{\beta}{(1-\beta)^2}$, $|\beta| < 1$

ii) $x[n] = \left(\frac{1}{4}\right)^n u[n-4]$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$= \sum_{n=4}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\Omega n} \quad \text{— 1M}$$

$$X(\Omega) = \sum_{n=4}^{\infty} \left(\frac{1}{4} e^{-j\Omega}\right)^n$$

wkt

$$\sum_{n=k}^{\infty} \beta^n = \frac{\beta^k}{(1-\beta)}, \ |\beta|<1$$

since, $\left|\frac{1}{4} e^{-j\Omega}\right| < 1$

$$X(\Omega) = \frac{\left(\frac{1}{4}\right)^4 e^{-j4\Omega}}{1-\frac{1}{4} e^{-j\Omega}} \quad \text{— 1M}$$

$$X(\Omega) = \left(\frac{1}{4}\right)^4 \frac{e^{-j4\Omega}}{1-\frac{1}{4}\left[\cos\Omega - j\sin\Omega\right]}$$

$$\therefore \boxed{X(\Omega) = \left(\frac{1}{4}\right)^4 \frac{e^{-j4\Omega}}{\left(1-\frac{1}{4}\cos\Omega\right) + j\left(\frac{1}{4}\sin\Omega\right)}} \quad \text{— 1M}$$

ii) $x[n] = \left(\frac{1}{4}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n]$

From convolution we know that

$$y[n] = \frac{1}{\beta-\alpha}\left[\beta^{n+1} - \alpha^{n+1}\right], \quad \alpha \neq \beta, \ n \geq 0 \quad \text{— 1M}$$

$$\alpha = \frac{1}{4} \ \& \ \beta = \frac{1}{3}$$

$$y[n] = 12\left[\frac{1}{3}\left(\frac{1}{3}\right)^n - \frac{1}{4}\left(\frac{1}{4}\right)^n\right].$$

$$\therefore y[n] = 4\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{4}\right)^n. \quad \text{— 1M}$$

$$Y(\Omega) = \sum_{n=0}^{\infty} y[n]\, e^{-j\Omega n}$$

$$Y(\Omega) = \sum_{n=0}^{\infty} \left[4\left(\tfrac{1}{3}\right)^n - 3\left(\tfrac{1}{4}\right)^n\right] e^{-j\Omega n}$$

$$Y(\Omega) = \sum_{n=0}^{\infty} 4\left(\tfrac{1}{3}\right)^n e^{-j\Omega n} - \sum_{n=0}^{\infty} 3\left(\tfrac{1}{4}\right)^n e^{-j\Omega n}$$

$$= 4\sum_{n=0}^{\infty}\left(\tfrac{1}{3}e^{-j\Omega}\right)^n - 3\sum_{n=0}^{\infty}\left(\tfrac{1}{4}e^{-j\Omega}\right)^n$$

$$Y(\Omega) = 4\,\frac{1}{1-\frac{1}{3}e^{-j\Omega}} - 3\,\frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

$\because \sum_{n=0}^{\infty} \beta^n = \frac{1}{1-\beta}, \; |\beta| <$

$$\therefore \boxed{Y(\Omega) = \frac{1}{\left(1-\frac{1}{3}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)}} \quad \text{— 1M}$$

Module - 5

9\. a) Find the Z-transform of the signal, $x[n] = (n(-0.5)^n u[n]) * 4^n u[-n]$.

[Total - 8M]

→ $x[n] = [n(-0.5)^n u[n]] * 4^n u[-n]$.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \text{— 1M}$$

$$X(z) = Z\{x[n]\} = Z\left\{n\left(-\tfrac{1}{2}\right)^n u[n]\right\} \cdot Z\left\{4^n u[-n]\right\} \quad \text{— 1M}$$

by using convolution property of Z-transform.

$$Z\left\{n\left(-\tfrac{1}{2}\right)^n u[n]\right\} = \sum_{n=-\infty}^{\infty} n\left(-\tfrac{1}{2}\right)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} n\left(-\tfrac{1}{2}z^{-1}\right)^n = \frac{-\frac{1}{2}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)^2} \quad \text{— 1M}$$

$\because \sum_{n=0}^{\infty} n\beta^n = \frac{\beta}{(1-\beta)^2}$

$$Z\{4^n u[-n]\} = \sum_{n=-\infty}^{0} 4^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} (0.25z)^n$$ 27

— 2M

$$\therefore Z\{4^n u[-n]\} = \frac{1}{1-\frac{z}{4}}$$ — 2M

$$\therefore X(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)^2} \cdot \frac{1}{\left(1-\frac{z}{4}\right)}$$ — 1M

Q.b) Using long division method, find the inverse z-transform of $X(z) = \frac{2+z^{-1}}{1-0.5z^{-1}}$, RoC: $|z| > 0.5$. [Total – 4M]

→ $X(z) = \frac{2+z^{-1}}{1-0.5z^{-1}}$

Long division method is performed as follows:

$$\begin{array}{r|l} & 2 + 2z^{-1} + z^{-2} + 0.5z^{-3} + 0.25z^{-4} \\ 1-0.5z^{-1} & \cancel{2} + z^{-1} \\ & -\cancel{2} \mp z^{-1} \\ & 2\cancel{z^{-1}} \\ & -\cancel{2z^{-1}} \mp z^{-2} \\ & \cancel{z^{-2}} \\ & -\cancel{z^{-2}} \mp 0.5z^{-3} \\ & 0.\cancel{5}z^{-3} \\ & -0.\cancel{5}z^{-3} \mp 0.25z^{-4} \\ & 0.2\cancel{5}z^{-4} \\ & -0.2\cancel{5}z^{-4} \mp 0.125z^{-5} \\ & 0.125z^{-5} \ldots \end{array}$$

— 2M

$$X(z) = \frac{2+z^{-1}}{1-0.5z^{-1}} = 2 + 2z^{-1} + z^{-2} + 0.5z^{-3} + 0.25z^{-4} + 0.125z^{-5} + \ldots$$ — 1M

$\therefore x[n] = [2, 2, 1, 0.5, 0.25, 0.125, \ldots]$. — 1M

9.c) An LTI system has impulse response, $h[n] = 0.5^n u[n]$.
Determine the input to the system if the output is given by, $y[n] = 0.5^n u[n] + (-0.5)^n u[n]$. [Total- 8M]

⇒ we know that, $H(z) = \frac{Y(z)}{X(z)}$

$$\therefore \quad X(z) = \frac{Y(z)}{H(z)} \quad \text{— 1M}$$

Z-transform of $h[n]$ & $y[n]$.

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad \& \quad Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{2}z^{-1}} \quad \text{— 2M}$$

$$Y(z) = \frac{2}{1-0.25z^{-2}} = \frac{2}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})} \quad \text{— 1M}$$

$$\therefore \quad X(z) = \frac{2}{(1-\cancel{\frac{1}{2}z^{-1}})(1+\frac{1}{2}z^{-1})} \cdot \frac{(1-\cancel{\frac{1}{2}z^{-1}})}{1} \quad \text{— 1M}$$

$$X(z) = 2 \frac{1}{1+\frac{1}{2}z^{-1}} = 2 \frac{1}{1-(-\frac{1}{2}z^{-1})} \quad \text{— 1M}$$

taking inverse z-transform we get.

$$\boxed{x[n] = 2\left(-\tfrac{1}{2}\right)^n u[n].} \quad \text{— 2M}$$

'OR'

10.a) Determine the transfer function and a difference equation representation of an LTI system described by the impulse response: $h[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-2} u[n-1]$.

[Total-8M]

$\rightarrow h[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-2} u[n-1].$

$h[n] = (0.33)^n u[n] + 4(0.5)^n u[n-1]$ — 1M

taking Z-transform,

$$H(Z) = \sum_{n=-\infty}^{\infty} (0.33)^n u[n] Z^{-n} + 4 \sum_{n=-\infty}^{\infty} (0.5)^n u[n-1] Z^{-n}$$ — 1M

$$H(Z) = \sum_{n=0}^{\infty} (0.33 Z^{-1})^n + 4 \sum_{n=1}^{\infty} (0.5 Z^{-1})^n$$

$$H(Z) = \frac{1}{1-0.33Z^{-1}} + \frac{2Z^{-1}}{1-0.5Z^{-1}}$$ — 1M $\left[\because \sum_{n=K}^{\infty} \beta^n = \frac{\beta^K}{1-\beta}, |\beta| < 1\right.$

$$\therefore \boxed{H(Z) = \frac{-0.66Z^{-2} + 1.5Z^{-1} + 1}{0.165Z^{-2} - 0.83Z^{-1} + 1}}$$ — 1M Transfer function.

we know that, $H(Z) = \frac{Y(Z)}{X(Z)}$ — 1M

$$\therefore \frac{Y(Z)}{X(Z)} = \frac{1}{1-0.33Z^{-1}} + \frac{2Z^{-1}}{1-0.5Z^{-1}}$$

$$Y(Z) = \frac{X(Z)}{1-0.33Z^{-1}} + \frac{X(Z)\, 2Z^{-1}}{1-0.5Z^{-1}}$$ — 1M

taking inverse Z-transform,

$$y[n] = \left(\frac{1}{3}\right)^n x[n] + 2\left(\frac{1}{2}\right)^{n-1} x[n-1]$$ — 1M

or

$$\boxed{y[n] = (0.33)^n x[n] + 2(0.5)^{n-1} x[n-1]}$$ — 1M

Difference equation.

10.b) A stable and causal LTI system is described by the difference equation: $y[n] + 0.25y[n-1] - 0.125y[n-2] = -2x[n] + 1.25x[n-1]$

Find the system impulse response. [Total - 8M]

$\rightarrow$ $y[n] + 0.25y[n-1] - 0.125y[n-2] = -2x[n] + 1.25x[n-1]$.

taking Z-transform.

$$Y(z) + 0.25z^{-1}Y(z) - 0.125z^{-2}Y(z) = -2X(z) + 1.25z^{-1}X(z).$$ — 1M

$$Y(z)[1 + 0.25z^{-1} - 0.125z^{-2}] = X(z)[-2 + 1.25z^{-1}].$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{-2 + 1.25z^{-1}}{1 + 0.25z^{-1} - 0.125z^{-2}}$$ — 1M

$$H(z) = \frac{-2z^2 + 1.25z}{z^2 + 0.25z - 0.125}$$ — 1M

$$\therefore \frac{H(z)}{z} = \frac{-2z + 1.25}{(z-0.25)(z+0.5)} = \frac{A}{z-0.25} + \frac{B}{z+0.5}$$ — 1M

$$A = (z-0.25)\frac{H(z)}{z}\Big|_{z=0.25} = \frac{-2z+1.25}{z+0.5}\Big|_{z=0.25} = 1$$ — 1M

$$B = (z+0.5)\frac{H(z)}{z}\Big|_{z=-0.5} = \frac{-2z+1.25}{z-0.25}\Big|_{z=-0.5} = -3$$

$$\therefore \frac{H(z)}{z} = \frac{1}{z-0.25} - \frac{3}{z+0.5}$$ — 1M

$$H(z) = \frac{1}{1-0.25z^{-1}} - \frac{3}{1+0.5z^{-1}}$$ — 1M taking inverse Z transform

$$\boxed{h[n] = [(0.25)^n - 3(-0.5)^n]u[n]}$$ — 1M

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10.c) Find the Z-transform of, $x[n] = 0.5^n u[n] + 2^n u[-n-1]$.

[Total - 4M]

→ $x[n] = 0.5^n u[n] + 2^n u[-n-1]$.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} [0.5^n u[n] + 2^n u[-n-1]] z^{-n} \quad \text{— 1M}$$

$$X(z) = \sum_{n=0}^{\infty} (0.5 z^{-1})^n + \sum_{n=-\infty}^{-1} (2z^{-1})^n$$

$$X(z) = \sum_{n=0}^{\infty} (0.5 z^{-1})^n + 1 - \sum_{k=0}^{\infty} (0.5 z)^k \quad \text{— 1M}$$

$$X(z) = \frac{1}{1-0.5z^{-1}} + 1 - \frac{1}{1-0.5z} \quad \text{— 1M}$$

$$\boxed{X(z) = \frac{z(z-1.25)}{(z-0.5)(0.5z-1)}} \quad \text{— 1M}$$