CBCS SCHEME

USN 18MAT21

Second Semester B.E. Degree Examination, Dec.2019/Jan.2020
Advanced Calculus and Numerical Methods

Time: 3 hrs. Max. Marks: 100

Note: Answer any FIVE full questions, choosing ONE full question from each module.

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

Module-1

1 a. Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at (2, -1, 2) along $2\bar{i} - 3\bar{j} + 6\bar{k}$. **(06 Marks)**

b. If $\bar{f} = \nabla(x^3 + y^3 + z^3 - 3xyz)$ find div $\bar{f}$ and curl $\bar{f}$. **(07 Marks)**

c. Find the constants a and b such that $\bar{F} = (axy + z^3)\bar{i} + (3x^2 - z)\bar{j} + (bxz^2 - y)\bar{k}$ is irrotational. Also find a scalar potential ϕ if $\bar{F} = \nabla\phi$. **(07 Marks)**

OR

2 a. If $\bar{F} = xy\bar{i} + yz\bar{j} + zx\bar{k}$ evaluate $\int_C \bar{F}.d\bar{r}$ where C is the curve represented by x = t, y = t^2, z = t^3, $-1 \le t \le 1$. **(06 Marks)**

b. Using Stoke's theorem Evaluate $\oint_C \bar{F}.d\bar{r}$ if $\bar{F} = (x^2 + y^2)\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by x = 0, x = a, y = 0, y = b. **(07 Marks)**

c. Using divergence theorem, evaluate $\iint_S \bar{F}.\hat{n}\,ds$ if $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ taken around $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. **(07 Marks)**

Module-2

3 a. Solve $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$ **(06 Marks)**

b. Solve $(D^2 + 4D + 3)y = e^{-3x}$ **(07 Marks)**

c. Using the method of variation of parameter solve $y'' + 4y = \tan 2x$. **(07 Marks)**

OR

4 a. Solve $(D^3 - 1)y = 3\cos 2x$ **(06 Marks)**

b. Solve $x^2y'' - 5xy' + 8y = 2\log x$ **(07 Marks)**

c. The differential equation of a simple pendulum is $\frac{d^2x}{dt^2} + W_0^2 x = F_0 \sin nt$, where W_0 and F_0 are constants. Also initially x = 0, $\frac{dx}{dt} = 0$ solve it. **(07 Marks)**

Module-3

5 a. Find the PDE by eliminating the function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$. **(06 Marks)**

b. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ given $\frac{\partial z}{\partial y} = -2\sin y$, when x = 0 and z = 0, when y is odd multiple of $\frac{\pi}{2}$. **(07 Marks)**

c. Derive one-dimensional wave equation in usual notations. **(07 Marks)**

OR

6 a. Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$ $\frac{\partial z}{\partial x} = a\sin y$ and $z = 0$. (06 Marks)

b. Solve $x(y - z)\, p + y\,(z - x)\, q = z\,(x - y)$. (07 Marks)

c. Find all possible solution of $U_t = C^2 U_{xx}$ one dimensional heat equation by variable separable method. (07 Marks)

Module-4

7 a. Test for convergence for
$1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \ldots\ldots\ldots$ (06 Marks)

b. Find the series solution of Legendre differential equation
$(1 - x^2)y'' - 2xy' + n(n + 1) = 0$ leading to $P_n(x)$. (07 Marks)

c. Prove the orthogonality property of Bessel's function as
$\int_0^1 x\, \bar{j}_n(\alpha x)\, \bar{j}_n(\beta x) dx = 0 \quad \alpha \neq \beta$ (07 Marks)

OR

8 a. Test for convergence for
$\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ (06 Marks)

b. Find the series solution of Bessel differential equation $x^2y'' + xy' + (n^2 - x^2)\, y = 0$ Leading to $\bar{j}_n(x)$. (07 Marks)

c. Express the polynomial $x^3 + 2x^2 - 4x + 5$ interms of Legendre polynomials. (07 Marks)

Module-5

9 a. Using Newton's forward difference formula find f(38). (06 Marks)

x	40	50	60	70	80	90
f(x)	184	204	226	250	276	304

b. Find the real root of the equation $x\log_{10} x = 1.2$ by Regula falsi method between 2 and 3 (Three iterations). (07 Marks)

c. Evaluate $\int_4^{5.2} \log x\, dx$ by Weddle's rule considering six intervals. (07 Marks)

OR

10 a. Find f(9) from the data by Newton's divided difference formula:

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

(06 Marks)

b. Using Newton – Raphson method, find the real root of the equation x sin x + cosx = 0 near $x = \pi$. (07 Marks)

c. By using Simpson's $\left(\frac{1}{3}\right)$ rule, evaluate $\int_0^6 \frac{dx}{1 + x^2}$ by considering seven ordinates. (07 Marks)

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Module - 1

1a) Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2i - 3j + 6k$

Sol? $\phi = 4xz^3 - 3x^2y^2z$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\nabla.\phi = (4z^3 - 6xy^2z)i + (-6x^2yz)j + (12xz^2 - 3x^2y^2)k$$

$$[\nabla\phi]_{(2,-1,2)} = 8i + 48j + 84k$$

The unit vector in the direction of $2i - 3j + 6k$ is

$$\hat{n} = \frac{2i - 3j + 6k}{\sqrt{4+9+36}} = \frac{2i - 3j + 6k}{\sqrt{49}} = \frac{2i - 3j + 6k}{7}$$

Thus

$$\nabla\phi \cdot \hat{n} = \frac{(8)(2) + (48)(-3) + (84)(6)}{7} = \frac{376}{7}$$

//

1 b) If $\vec{f} = \nabla(x^3 + y^3 + z^3 - 3xyz)$ find div $\bar{f}$ and curl $\bar{f}$

Sol: $\phi = x^3 + y^3 + z^3 - 3xyz$

$$\vec{f} = \nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\vec{f} = (3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k$$

$$\text{div}\,\bar{f} = \nabla \cdot \vec{f}$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\left([3x^2 - 3yz]i + [3y^2 - 3xz]j + [3z^2 - 3xy]k\right)$$

$$\text{div}\,\vec{f} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$\text{div}\,\vec{f} = 6x + 6y + 6z = 6(x+y+z)$$

$$\text{div}\,\vec{f} = 6(x+y+z)$$

$$\text{curl}\,\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$\text{curl } \vec{f} = i\left[-3x - (-3x)\right] - j\left[-3y - (-3y)\right]$$

$$+ k\left[-3z - (-3z)\right]$$

$$\text{curl } \vec{f} = i\left[\cancel{-3x} + \cancel{3x}\right] - j\left[\cancel{-3y} + \cancel{3y}\right] +$$

$$k\left[\cancel{-3z} + \cancel{3z}\right]$$

$$\text{curl } \vec{f} = i(0) - j(0) + k(0)$$

$$\text{curl } \vec{f} = 0.$$

1c) Find the consts a and b such that $\vec{F} = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational. Also find a scalar potential ϕ if $\vec{F} = \nabla\phi$

Sol?. we have to find a and b such that

curl $\vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy+z^3) & (3x^2-z) & bxz^2-y \end{vmatrix} = 0$$

$$i\{-1-(-1)\} - j\{bz^2 - 3z^2\} + k\{6x - ax\} = 0$$

$$i(\cancel{-1}+\cancel{1}) - j\{b-3\}z^2 + k(6-a)x = 0$$

$$-z^2(b-3)j + x(6-a)k = 0$$

The above Eqn is identically satisfied when $b-3=0$ and $6-a=0$

$\Rightarrow$ $b = 3$ and $a = 6$

Now consider, $\nabla\phi = \vec{F}$ when $a=6$ $b=3$

$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (6xy+z^3)i + (3x^2-z)j + (3xz^2-y)k$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 6xy + z^3$$

$$\Rightarrow \phi = \int (6xy + z^3)\, dx + f_1(y, z)$$

$$\phi = 3x^2y + xz^3 + f_1(y, z) \quad \text{——— (1)}$$

$$\frac{\partial \phi}{\partial y} = (3x^2 - z) \Rightarrow \phi = \int (3x^2 - z)\, dy + f_2(x, z)$$

$$\phi = 3x^2y - yz + f_2(z) \quad \text{——— (2)}$$

$$\frac{\partial \phi}{\partial z} = (3xz^2 - y) \Rightarrow \phi = \int (3xz^2 - y)\, dz + f_3(x, y)$$

$$\phi = xz^3 - yz + f_3(x, y) \quad \text{——— (3)}$$

$f_1(y, z) = -yz \qquad f_2(x, z) = xz^3$

$f_3(x, y) = 3x^2y$

Thus required $\phi = 3x^2y - yz + xz^3$ //

2a) If $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ Evaluate $\int_C \vec{F} \, dr$

where c is curve represented by

$x = t$, $y = t^2$, $z = t^3$, $-1 \leq t \leq 1$

Soln. we have $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$

$$\vec{r} = xi + yj + zk$$

$$\Rightarrow \quad d\vec{r} = dx\,i + dy\,j + dz\,k$$

$$\vec{F}\,dr = xy\,dx + yz\,dy + zx\,dz$$

since $x = t$ $y = t^2$ $z = t^3$ by data

$dx = dt$ $dy = 2t\,dt$ $dz = 3t^2\,dt$

$$\vec{F}\,dr = t^3\,dt + t^5(2t)\,dt + t^4(3t^2)\,dt$$

$$= (t^3 + 2t^6 + 3t^6)\,dt$$

$$\vec{F}\,dr = (t^3 + 5t^6)\,dt$$

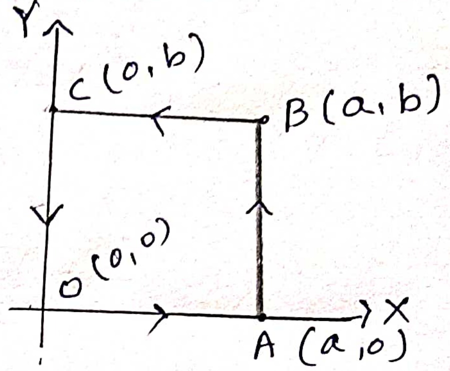
$$\int_C \vec{F}\,dr = \int_{-1}^{1} (t^3 + 5t^6)\,dt$$

$$= \left[\frac{t^4}{4} + 5\frac{t^7}{7}\right]_{-1}^{1}$$

$$= \frac{1}{4}\{\cancel{1} - \cancel{1}\} + \frac{5}{7}\{1 - (-1)\} = \frac{10}{7}$$

$$\int_C \vec{F}\,dr = 10/7$$

2b) Using stoke's theorem Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ if $\vec{F} = (x^2+y^2)i - 2xyj$ taken round the rectangle bounded by $x=0$, $x=a$ $y=0$, $y=b$.



Sol: we have stoke's theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$$

Now $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$

$$= i(0-0) - j(0-0) + k(-2y-2y)$$

$$\nabla \times \vec{F} = -4yk$$

$$(\nabla \times \vec{F}) \cdot \hat{n}\, ds = (-4yk)(dy\,dz\, i + dz\,dx\, j + dx\,dy\, k)$$

$$= -4y\, d\ dy$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_R -4y\, dx\, dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{b} -4y\, dy\, dx$$

$$\int_C \vec{F}\,\vec{dr} = \int_{x=0}^{a}\left[-\overset{2}{\cancel{4}}\,\frac{y^2}{\cancel{2}}\right]_0^b dx$$

$$= -2\int_{x=0}^{a}\left[y^2\right]_0^b dx$$

$$= -2\int_{x=0}^{a}\left[b^2-0\right]dx.$$

$$= -2\int_{x=0}^{a} b^2\,dx \quad = -2b^2\int_{x=0}^{a} 1\,dx$$

$$= -2b^2\left[x\right]_0^a \quad = -2b^2\left[a-0\right]$$

$$\int_C \vec{F}\,\vec{dr} = -2ab^2.$$

2c) Using divergence theorem Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ if $\vec{F} = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$

taken around $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

Sol: We have $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv$

$$\vec{F} = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$\operatorname{div}\vec{F} = 2x + 2y + 2z = 2(x+y+z)$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 2(x+y+z)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2\left[xz + yz + \frac{z^2}{2}\right]_{z=0}^{1} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left\{2x[1-0] + 2y[1-0] + [1-0]\right\} \cdot dy\,dx$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1} (2x+2y+1)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[2xy + \cancel{2}\frac{y^2}{\cancel{2}} + y\right]_{y=0}^{1} dx$$

$$= \int_{x=0}^{1} \{2x[1-0] + [1-0] + [1-0]\}\,dx$$

$$= \int_{x=0}^{1} (2x+1+1)\,dx = \int_{x=0}^{1} (2x+2)\,dx$$

$$= 2\int_{x=0}^{1} (x+1)\,dx = 2\left[\frac{x^2}{2} + x\right]_{x=0}^{1}$$

$$= 2\left[\left(\frac{1}{2}+1\right) - 0\right] = \cancel{2}\left[\frac{3}{\cancel{2}}\right]$$

$$= 3$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 3$$

3(a) Solve $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$

Sol: A.E is $4m^4 - 8m^3 - 7m^2 + 11m + 6 = 0$

$$\begin{array}{r|rrrrr} -1 & 4 & -8 & -7 & 11 & \cancel{6} \\ & - & -4 & +12 & -5 & -\cancel{6} \\ \hline & 4 & -12 & 5 & 6 & 0 \end{array} \qquad m = -1.$$

$$\begin{array}{r|rrrr} 2 & 4 & -12 & 5 & \cancel{6} \\ & - & 8 & -8 & -\cancel{6} \\ \hline & 4 & -4 & -3 & 0 \end{array} \qquad m = 2$$

$4m^2 - 4m - 3 = 0$

$4m^2 - 6m + 2m - 3 = 0$

$2m(2m-3) + 1(2m-3) = 0$

$(2m-3)(2m+1) = 0$

$\Rightarrow m = -\frac{1}{2}, \frac{3}{2}$

Thus we have $m = -\frac{1}{2}, \frac{3}{2}, -1, 2.$

Hence solution is

$y(x) = c_1 e^{-x/2} + c_2 e^{3x/2} + c_3 e^{-x} + c_4 e^{2x}$

3 b) Solve $(D^2+4D+3)y = e^{-x}$

Sol? A.E is $m^2+4m+3=0$

$m^2+3m+m+3=0 \Rightarrow m(m+3)+1(m+3)=0$

$(m+1)(m+3)=0 \Rightarrow m=-1,-3$

$y_c(x) = c_1 e^{-x} + c_2 e^{-3x}$

$y_p(x) = \frac{X}{f(D)} = \frac{e^{-x}}{D^2+4D+3}$ Put $D=-1$ in $f(D)$

$y_p(x) = \frac{e^{-x}}{(-1)^2+4(-1)+3} = \frac{e^{-x}}{1-4+3} = \frac{e^{-x}}{\cancel{4}-\cancel{4}} = \frac{e^{-x}}{0}$

$y_p(x) = \frac{x e^{-x}}{f'(D)} = \frac{x e^{-x}}{2D+4}$ Put $D=-1$ in $f'(D)$

$y_p(x) = \frac{x e^{-x}}{2(-1)+4} = \frac{x e^{-x}}{-2+4} = \frac{x e^{-x}}{2}$

Hence G.S = C.F + P.I

$y(x) = y_c(x) + y_p(x)$

$y(x) = c_1 e^{-x} + c_2 e^{-3x} + \frac{x e^{-x}}{2}$

3(c) Using method of variation of parameter solve $y'' + 4y = \tan 2x$

Sol^n. We have $y'' + 4y = \tan 2x$

$$y' = \frac{dy}{dx} \Rightarrow y'' = \frac{d^2y}{dx^2} \quad \text{Put } D = \frac{d}{dx}$$

$$\Rightarrow y' = Dy \Rightarrow y'' = D^2 y$$

$$\Rightarrow D^2 y + 4y = \tan 2x$$

$$(D^2 + 4)y = \tan 2x$$

A.E is $m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m^2 = (-1)4$

$$m^2 = i^2(4) \Rightarrow m^2 = (2i)^2 \Rightarrow m = \pm 2i$$

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

compairing with $y_c(x) = c_1\, y_1(x) + c_2\, y_2(x)$

$\Rightarrow \quad y_1(x) = \cos 2x \qquad y_2(x) = \sin 2x$

Let us assume that

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

where $u_1(x) = -\int \frac{X\, y_2(x)}{W}\, dx$

$$u_1(x) = \int \frac{X\, y_1(x)}{W}\, dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$W = 2\cos^2 2x - (-2\sin^2 2x)$$

$$W = 2\{\cos^2 2x + \sin^2 2x\} = 2 \times 1 = 2$$

$$\boxed{W = 2}$$

$$u_1(x) = -\int \frac{\tan 2x \ \sin 2x}{2}\, dx$$

$$= -\frac{1}{2}\int \frac{\sin 2x \cdot \sin 2x}{\cos 2x}\, dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\, dx$$

$$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\, dx$$

$$= -\frac{1}{2}\int\left\{\frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x}\right\} dx$$

$$= -\frac{1}{2}\int\{\sec 2x - \cos 2x\}\, dx$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$u_1(x) = -\frac{1}{4}\left\{\log(\sec 2x + \tan 2x) - \sin 2x\right\}$$

$$u_2(x) = \int \frac{\tan 2x \cdot \cos 2x}{2}\, dx$$

$$u_2(x) = \int \frac{\sin 2x}{\cancel{\cos 2x}} \cdot \frac{\cancel{\cos 2x}}{2}\, dx$$

$$u_2(x) = \frac{1}{2} \int \sin 2x \, dx = \frac{1}{2} \left\{ -\frac{\cos 2x}{2} \right\}$$

$$u_2(x) = -\frac{1}{4} \cos 2x$$

$$y_p(x) = -\frac{1}{4} \left\{ \log(\sec 2x + \tan 2x) - \sin 2x \right\} \cos 2x + \left\{ -\frac{1}{4} \cos 2x \right\} \sin 2x$$

$$y_p(x) = -\frac{1}{4} \left\{ \log(\sec 2x + \tan 2x) \cdot \cos 2x - \cancel{\sin 2x \cos 2x} + \cancel{\sin 2x \cos 2x} \right\}$$

$$y_p(x) = -\frac{1}{4} \left\{ \log(\sec 2x + \tan 2x) \cdot \cos 2x \right\}$$

G.S = C.F + P.I

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \left\{ -\frac{1}{4} \left[\log(\sec 2x + \tan 2x) \cdot \cos 2x \right] \right\}$$

$$y(x) = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \left[\log(\sec 2x + \tan 2x) \cdot \cos 2x \right]$$

4(a) Solve $(D^3 - 1)y = 3\cos 2x$

Sol^n A.E $m^3 - 1 = 0 \Rightarrow (m-1)(m^2+m+1) = 0$

$m = 1 \quad m^2 + m + 1 = 0 \Rightarrow m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$m = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$y_c(x) = c_1 e^x + e^{-x/2}\left\{c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right\}$

$y_p(x) = \frac{X}{f(D)} = \frac{3\cos 2x}{D^3 - 1} = 3\frac{\cos 2x}{D^2 \cdot D - 1}$.

Put $D^2 = -2^2 = -4$ in $f(D)$

$y_p(x) = 3\frac{\cos 2x}{(-4)D - 1} = 3\frac{\cos 2x}{-4D - 1}$

$y_p(x) = 3\frac{\cos 2x}{(-4D-1)} \times \frac{(-4D+1)}{(-4D+1)}$

$y_p(x) = 3\left\{\frac{-4D\cos 2x + \cos 2x}{(-4D)^2 - (1)^2}\right\}$

$y_p(x) = 3\left\{\frac{(-4)(-\sin 2x)(2) + \cos 2x}{16D^2 - 1}\right\}$

Put $D^2 = -2^2 = -4$

$$y_p(x) = 3\left\{\frac{8\sin 2x + \cos 2x}{16(-4)-1}\right\}$$

$$= \frac{3}{-65}\left\{8\sin 2x + \cos 2x\right\}$$

$$y_p(x) = -\frac{3}{65}\left\{8\sin 2x + \cos 2x\right\}$$

$$G.S = C.F + P.I$$

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 e^{x} + e^{-x/2}\left\{c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right\}$$

$$- \frac{3}{65}\left\{8\sin 2x + \cos 2x\right\}$$

//

4(b) Solve $x^2 y'' - 5xy' + 8y = 2\log x$

Sol. This is Cauchy's linear Equation

Put $x = e^t \Rightarrow t = \log x$. If $D = \frac{d}{dt}$ then

$xy' = Dy \Rightarrow x^2 y'' = D(D-1)y$

substituting in above Eqn

$$D(D-1)y - 5Dy + 8y = 2t$$

$$(D^2 - D - 5D + 8)y = 2t$$

$$(D^2 - 6D + 8)y = 2t$$

A E is $m^2 - 6m + 8 = 0$

$(m-2)(m-4) = 0 \Rightarrow m = 2, 4$

$$y_c(t) = c_1 e^{2t} + c_2 e^{4t}$$

$$y_p(t) = \frac{2t}{D^2 - 6D + 4} = 2\,\frac{t}{4 - 6D + D^2}$$

$$
\begin{array}{r|l}
 & t/4 + 3/8 \\
\hline
4 - 6D + D^2 & \cancel{t} \\
 & \cancel{t} - 3/2 + 0 \\
 & (-)\quad t \\
\hline
 & \cancel{3/2} \\
 & 3/2 - 0 + 0 \\
 & (-) \\
\hline
 & 0
\end{array}
$$

$$\therefore\ y_p(t) = 2\left(\frac{t}{4} + \frac{3}{8}\right) = \frac{t}{2} + \frac{3}{4}$$

$G.S = C.F + P.I$

$y(t) = y_c(t) + y_p(t)$

$y(t) = c_1 e^{2t} + c_2 e^{4t} + \left(\frac{t}{2} + \frac{3}{4}\right)$

Put $t = \log x$

$e^t = x \Rightarrow e^{2t} = x^2, \quad e^{4t} = x^4$

$\Rightarrow y(x) = c_1 x^2 + c_2 x^4 + \frac{\log x}{2} + \frac{3}{4}$

or $y(x) = c_1 x^2 + c_2 x^4 + \frac{1}{2}\left[\log x + \frac{3}{2}\right]$

is required solution

4(c) The differential Equation of a simple pendulum is $\frac{d^2x}{dt^2} + W_0 x = F_0 \sin nt$, where W_0 and F_0 are constants. Also initially $x = 0$, $\frac{dx}{dt} = 0$ and take $n = W_0$, Solve it

Sol: Put $D = \frac{d}{dt} \Rightarrow D^2 = \frac{d^2}{dt^2}$

$\Rightarrow (D^2 + W_0)x = F_0 \sin nt$

A.E is $D^2 + W_0^2 = 0 \Rightarrow D^2 = -W_0^2 = (-1)W_0^2$

$D^2 = W_0^2 i^2 = (iW_0)^2 \Rightarrow D = \pm iW_0$

$x_c(t) = C_1 \cos W_0 t + C_2 \sin W_0 t$

$P.I = x_p(t) = \frac{F_0 \sin nt}{D^2 + W_0^2}$

Put $D^2 = -n^2$
but given that
$n = W_0$, $D^2 = -W_0^2$

$x_p(t) = \frac{F_0 \sin W_0 t}{-W_0^2 + W_0^2} = \frac{F_0 \sin W_0 t}{0}$

$x_p(t) = F_0 . t \frac{\sin W_0 t}{f'(D)} = F_0 t \frac{\sin W_0 t}{2D}$

$$x_p(t) = F_0 \frac{t}{2}\frac{1}{D}\sin W_0 t = \frac{F_0 t}{2}\left\{-\frac{\cos W_0 t}{W_0}\right\}$$

$$x_p(t) = -\frac{F_0 t}{2W_0}\cos W_0 t$$

$G.S = C.F + P.I$

$$\therefore x(t) = c_1 \cos W_0 t + C_2 \sin W_0 t - \frac{F_0 t}{2W_0}\cos W_0 t \quad \text{—— (1)}$$

Differentiate Eqn (1) w.r.t t

$$x'(t) = -C_1 W_0 \sin W_0 t + C_2 W_0 \cos W_0 t - \left\{\frac{F_0 t}{2\cancel{W_0}}[-\sin W_0 t \cdot \cancel{W_0}] + \cos W_0 t \cdot \frac{F_0}{2W_0}\right\}$$

$$x'(t) = -W_0 C_1 \sin W_0 t + W_0 C_2 \cos W_0 t + \frac{F_0 t}{2}\sin W_0 t - \frac{F_0}{2W_0}\cos W_0 t \quad \text{—— (2)}$$

Given that $x(0) = 0$ $x'(0) = 0$

Using these conditions in Eqn (1) and Eqn (2) we get

$x(0) = C_1 + 0 - 0 \Rightarrow 0 = C_1$

$\therefore C_1 = 0$

$$x'(0) = -0 + W_0 C_2 + 0 - \frac{F_0}{2W_0}$$

$$0 = W_0 C_2 - \frac{F_0}{2W_0}$$

$$\Rightarrow C_2 W_0 = \frac{F_0}{2W_0} \Rightarrow C_2 = \frac{F_0}{2W_0^2}$$

Substituting in Eqn (1)

$$x(t) = (0) + \frac{F_0}{2W_0^2} \sin W_0 t - \frac{F_0 t}{2W_0} \cos W_0 t$$

$$x(t) = \frac{F_0}{2W_0} \left\{ \frac{1}{W_0} \sin W_0 t - t \cos W_0 t \right\}$$

which is required solution.

5 a) Find the PDE by Eliminating the function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

sol? $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ —— (1)

Differentiate w.r.t x and w.r.t y partially

$\frac{\partial z}{\partial x} = 0 + 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$

$\frac{\partial z}{\partial x} = -\frac{2}{x^2} f'\left(\frac{1}{x} + \log y\right)$ —— (2)

$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$

$y\frac{\partial z}{\partial y} = 2y^2 + 2f'\left(\frac{1}{x} + \log y\right)$

$qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right)$ —— (3)

Using Eqn (3) in Eqn (2)

$p = \frac{-1}{x^2}\{qy - 2y^2\}$

$px^2 = -qy + 2y^2 \Rightarrow px^2 = 2y^2 - qy$

$px^2 = y(2y - q)$ or.

$\frac{\partial z}{\partial x} x^2 = 2y^2 - \frac{\partial z}{\partial y} y$ or

$\frac{\partial z}{\partial x} x^2 + \frac{\partial z}{\partial y} y = 2y^2$ is required P.D.E

5 b) Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ given $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$ and $z=0$ when y is odd multiple of $\pi/2$ (ie $y = (2n+1)/2$)

Sol^n. $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t x treating y as constant

$$\frac{\partial z}{\partial y} = -\sin y \cos x + f(y) \quad \text{—— (1)}$$

Integrating w.r.t y treating x as const

$$z = -\cos x(-\cos y) + \int f(y)\,dy + g(x)$$

$$z = \cos x \cos y + F(y) + g(x) \quad \text{—— (2)}$$

where $F(y) = \int f(y)\,dy$

Given that $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$

using this in Eqn (1)

$$-2\sin y = -\sin y \cos 0 + f(y)$$

$$-2\sin y = -\sin y + f(y)$$

$$f(y) = -2\sin y + \sin y = -\sin y$$

$$f(y) = -\sin y$$

$F(y) = \int f(y)\,dy \Rightarrow F(y) = \int (-\sin y)\,dy$

$F(y) = -(-\cos y) = \cos y \Rightarrow \boxed{F(y) = \cos y}$

Given that $z = 0$ if $y = (2n+1)\pi/2$.

Using this in Eqn (2) we get

$$0 = \cos x \left[\cos (2n+1)\frac{\pi}{2}\right] + \cos y + g(x)$$

$\because F(y) = \cos y$

$$0 = \cos x \left[\cos (2n+1)\pi/2\right] + \left[\cos (2n+1)\frac{\pi}{2}\right] + g(x)$$

But $\cos (2n+1)\frac{\pi}{2} = 0$

$\Rightarrow 0 = 0 + 0 + g(x) \Rightarrow g(x) = 0$

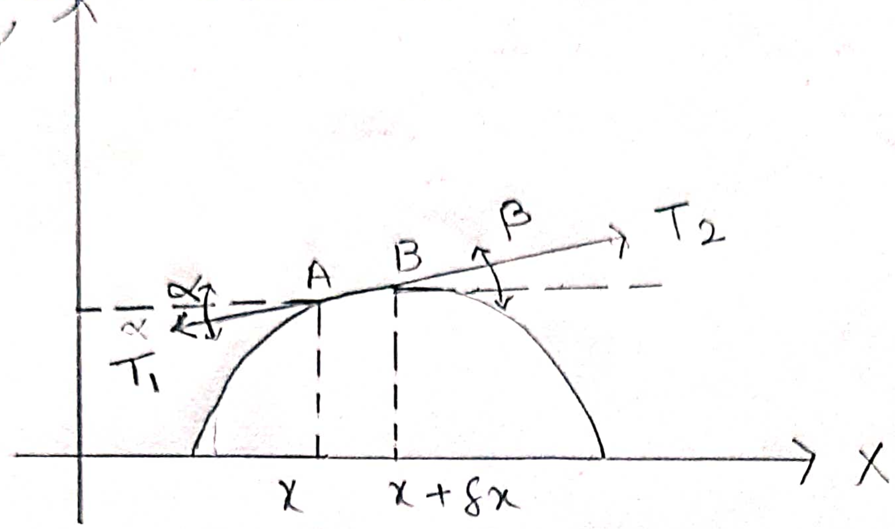
Thus the solution of PDE is

$z = \cos x \cos y + \cos y + 0$ $\quad \because g(x) = 0$

$z = \cos y (\cos x + 1)$

5 c) Derive one dimensional wave Eqn in usual notations.

Sol:



consider a flexible string tightly stretched between two fixed points at a distance l apart. let ρ be mass per unit length of string. we shall assume that

(i) Tension T of the string is same throghout.

(ii) The effect of gravity can be ignored due to a large tension T

(iii) The motion of the string is in small transverse vibrations

Let us consider forces acting on a small element AB of length δx. let T_1 and T_2 the tensions at the points A and B.

since there is no motion in horizontal direction, The horizontal Components

T_1 and T_2 must cancel each other.

$\therefore$ $T_1 \cos\alpha = T_2 \cos\beta = T$ —— (1)

where α and β are angles made by T_1 and T_2 with the horizontal, vertical components of tension are $-T_1 \sin\alpha$ and $T_2 \sin\beta$ where -ve sign is used $\because$ T_1 is directed downwords. Hence the resultant force acting vertically upwards is $T_2 \sin\beta - T_1 \sin\alpha$

Applying Newton's 2nd law of motion

ie Force = mass x Acceleration

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho\, \delta x) \frac{\partial^2 u}{\partial t^2}$$

Dividing through out by T, we have

$$\frac{T_2}{T} \sin\beta - \frac{T_1}{T} \sin\alpha = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}$$

But from Eqn (1) $\frac{T_1}{T} = \frac{1}{\cos\alpha}$ $\frac{T_2}{T} = \frac{1}{\cos\beta}$

$$\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{\rho}{T} \delta x \frac{\partial^2 u}{\partial t^2} \quad \text{—— (2)}$$

$\tan\beta$ and $\tan\alpha$ represents the slopes at $B(x+\delta x)$ and $A(x)$ respectively.

$$\tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \qquad \tan\alpha = \left(\frac{\partial u}{\partial x}\right)_{x}$$

Now Eqn (2) becomes

$$\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 u}{\partial t^2}$$

Dividing through out by δx taking limits as $\delta x \to 0$ we have

$$\lim_{\delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x} = \frac{\rho}{T}\,\frac{\partial^2 u}{\partial t^2}.$$

But the LHS is nothing but the derivative of $\frac{\partial u}{\partial x}$ w.r.t x, treating t as constant.

ie $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ or $\frac{\partial^2 u}{\partial x^2}$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\,\frac{\partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\,\frac{\partial^2 u}{\partial x^2}$$

Put $\frac{T}{\rho} = c^2$ we get $\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.}$

or $\boxed{u_{tt} = c^2 u_{xx}}$ this is wave Eqn in one dimensional.

5(a) Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$

$\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$

Sol?. Let us suppose that z is a function of x only. The given PDE assumes the form of ODE

$\frac{d^2 z}{dx^2} = a^2 z \Rightarrow D^2 z = a^2 z \Rightarrow (D^2 - a^2) z = 0$

A.E is $D^2 - a^2 = 0 \Rightarrow D^2 = a^2 \Rightarrow D = \pm a$

Hence solution is $z = c_1 e^{ax} + c_2 e^{-ax}$

solution of PDE is given by

$z = f(y) e^{ax} + g(y) e^{-ax}$ ——— (A)

Diff^{ate} partially w.r.t x and w.r.t y

$\frac{\partial z}{\partial x} = f(y)\, a e^{ax} + g(y)(-a) e^{-ax}$ ——— (1)

$\frac{\partial z}{\partial y} = e^{ax} f'(y) + e^{-ax} g'(y)$ ——— (2)

By data $\frac{\partial z}{\partial x} = a \sin y$ when $x = 0$

Hence Eqn (1) becomes

$a \sin y = a f(y) \div a g(y)$

$\cancel{a} \sin y = \cancel{a} [f(y) - g(y)]$

$\sin y = f(y) - g(y)$ ——— (3)

By data $\frac{\partial z}{\partial y} = 0$ when $x = 0$

Hence Eqn (2) becomes

$0 = f'(y) + g'(y)$ Integrating w.r.t y we get.

$f(y) + g(y) = k$ ——— (4) where k is const of integration.

solving Eqn (3) and Eqn (4)

$f(y) = \frac{1}{2}[k + \sin y]$

$g(y) = \frac{1}{2}[k - \sin y]$

substituting in Eqn (A) we get

$z = \frac{1}{2}[k + \sin y] e^{ax} + \frac{1}{2}[k - \sin y] e^{-ax}$.

6b) Solve $x(y-z)p + y(z-x)q = z(x-y)$

Sol: $x(y-z)p + y(z-x)q = z(x-y)$

This Eqn is in the form $Pp + Qq = R$

where $P = x(y-z)$ $Q = y(z-x)$ $R = z(x-y)$

Auxiliary Eqns are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \quad \text{——— (1)}$$

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers and adding

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z)+(z-x)+(x-y)} = 0$$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$ Intigrating

$\log x + \log y + \log z = \log a$

$\log xyz = \log a \Rightarrow xyz = a$ ——— (2)

Again adding we obtain

$$\frac{dx + dy + dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{xy} + \cancel{xz} - \cancel{yz}} = 0$$

$dx + dy + dz = 0$ Intigrating

$x + y + z = b$ ——— (3) G.S is $\phi(u, v) = 0$

ie $\phi(xyz, x+y+z) = 0$

6c) Find all possible solution of $u_t = c^2 u_{xx}$, one dimensional heat Eqn by variable separable method.

Sol: consider $u_t = c^2 u_{xx}$ ie $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

let $u = XT$ where $X = X(x)$ $T = T(t)$ be the soln of PDE. Hence PDE becomes

$$\frac{\partial (XT)}{\partial t} = c^2 \frac{\partial^2 (XT)}{\partial x^2}$$

$$X \frac{dT}{dt} = c^2 T \frac{d^2X}{dx^2} \quad \text{or} \quad \frac{1}{c^2T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2X}{dx^2}$$

Equating both sides to a common const K

$$\frac{1}{c^2T} \frac{dT}{dt} = K \Rightarrow \frac{dT}{dt} = c^2KT.$$

$$\Rightarrow DT = c^2KT \Rightarrow (D - c^2K)T = 0 \quad \text{——— (1)} \quad D = \frac{d}{dt}$$

$$\frac{1}{X} \frac{d^2X}{dx^2} = K \Rightarrow \frac{d^2X}{dx^2} = KX.$$

$$\Rightarrow D^2X = KX \Rightarrow (D^2 - K)X = 0 \quad \text{——— (2)} \quad D = \frac{d}{dx}$$

case (i) when $K = 0$ Eqn (1) becomes

$(D - 0)T = 0 \Rightarrow DT = 0 \Rightarrow \frac{dT}{dt}$. Integrating

$T = c_1$.

Putting $k=0$ in Eqn (2) $D^2 X = 0$

$\frac{d^2X}{dx^2} = 0$ Intigrating w.r.t x

$\Rightarrow \frac{dX}{dx} = C_2$ Intigrating w.r.t x again

$X = C_2 x + C_3$

Hence solution of PDE when $k=0$ is

$u = XT$, $u = C_1 [C_2 x + C_3]$

Case (ii) when $k = +p^2$ (Positive)

Eqn (1) becomes $(D - c^2p^2) \cdot T = 0$

$DT = c^2 p^2 T \Rightarrow \frac{dT}{dt} = c^2p^2T$ separating variables

$\frac{1}{T} dT = c^2p^2 dt$ Integrating

$\log T = c^2p^2t + c_1 \Rightarrow T = e^{c^2p^2t + c_1}$

$T = e^{c^2p^2t} \cdot e^{c_1} \Rightarrow T = c_1' e^{c^2p^2t}$

where $c_1' = e^{c_1}$

Eqn (2) becomes $(D^2 - p^2)X = 0$

A.E is $D^2 - p^2 = 0 \Rightarrow D = \pm p \Rightarrow$

$X = c_2' e^{px} + c_3' e^{-px}$

Hence solution of PDE when $k = p^2$ is

$u = XT$, $u = c_1' e^{c^2p^2t} [c_2' e^{px} + c_3' e^{-px}]$

case (iii) when $k = -p^2$ (negative)

Eqn (1) becomes $(D + c^2p^2)\,T = 0$

$\Rightarrow \frac{dT}{dt} = -c^2p^2T$ separating the variables

$\frac{1}{T}\,dT = -c^2p^2dt$ Intigrating

$\log T = -c^2p^2t + c_1$, $T = e^{-c^2p^2t + c_1}$

$T = e^{-c^2p^2t} \cdot e^{c_1} \Rightarrow T = c_1'' e^{-c^2p^2t}$

where $c_1'' = e^{c_1}$.

Eqn (2) becomes $(D^2 + p^2)\,X = 0$

AE is $D^2 + p^2 = 0 \Rightarrow D^2 = -p^2 = (-1)\,p^2$.

$D^2 = i^2p^2 \Rightarrow D^2 = (ip)^2 \Rightarrow D = \pm ip$

$X = c_2'' \cos px + c_3'' \sin px$

Hence solution of PDE when $k = -p^2$ is

$u = XT$, $u = (c_2'' \cos px + c_3'' \cos px)\, c_1'' e^{-c^2p^2t}$

Module - 4

7(a) Test for convergence for

$$1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \cdots$$

Sol^n: let $\Sigma u_n = 1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \cdots$

Here $u_n = \frac{n!}{n^2} \Rightarrow u_{n+1} = \frac{(n+1)!}{(n+1)^2}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^2} \cdot \frac{n^2}{n!} = \frac{(n+1)\cancel{n!}\, n^2}{(n+1)^2\, \cancel{n!}}$$

$$\frac{u_{n+1}}{u_n} = \frac{n\cdot\cancel{\left(1+\frac{1}{n}\right)}\,\cancel{n^2}}{\cancel{n^2}\left(1+\frac{1}{n}\right)^{\cancel{2}}} = \frac{n}{\left(1+\frac{1}{n}\right)}$$

Now $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{n}{1+\frac{1}{n}} = \frac{\infty}{1+0} = \infty > 1$

Hence by D'Alembert's Ratio Test

the series Σu_n is divergent.

7b) Find the series sol^n Legendre differential Equation $(1-x^2)y'' - 2xy' + n(n+1) = 0$ leading to $P_n(x)$

Sol^n. $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ ——— (1)

where $P_0(x) = 1 - x^2 \Rightarrow P_0(x) \neq 0$ at $x = 0$

Now assume $y = \sum_{r=0}^{\infty} a_r x^r$

$$y' = \sum_{r=0}^{\infty} a_r \, r \, x^{r-1}$$

$$y'' = \sum_{r=0}^{\infty} a_r \, r(r-1) \, x^{r-2}$$

substituting in Eqn (1)

$$(1-x^2)\left\{\sum_{r=0}^{\infty} a_r \, r(r-1) \, x^{r-2}\right\} - 2x\left\{\sum_{r=0}^{\infty} a_r \, r \, x^{r-1}\right\} + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\sum_{r=0}^{\infty} a_r \, r(r-1) \, x^{r-2} - \sum_{r=0}^{\infty} a_r \, r(r-1) \, x^{r} - 2\sum_{r=0}^{\infty} a_r \, r \, x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

Equating the coefficient of x^r

$$a_{r+2}(r+2)(r+1) - a_r\, r(r-1) - 2a_r r + n(n+1)a_r = 0$$

$$a_{r+2}(r+1)(r+2) - a_r[r^2 - r + 2r - n(n+1)] = 0$$

$$a_{r+2}(r+1)(r+2) = a_r[r^2 + r - n(n+1)]$$

$$a_{r+2} = \frac{a_r[r^2 + r - n(n+1)]}{(r+1)(r+2)}$$

Put $r = 0 \Rightarrow a_2 = \dfrac{a_0[-n(n+1)]}{1 \cdot 2}$

$$\Rightarrow a_2 = -\frac{n(n+1)}{2!}a_0$$

Put

$r = 1 \Rightarrow a_3 = \dfrac{a_1[2 - n^2 - n]}{2 \cdot 3} = \dfrac{-a_1[n^2 + n - 2]}{3!}$

$$a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

Put $r = 2 \Rightarrow a_4 = \dfrac{a_2[6 - n(n+1)]}{3 \cdot 4}$

$$a_4 = \frac{[6 - n^2 - n]}{3 \cdot 4}\left\{\frac{-a_0\, n(n+1)}{1 \cdot 2}\right\}$$

$$a_4 = \frac{-\{n^2+n-6\}}{3\cdot 4} \; \frac{\{-a_0 n(n+1)\}}{1\cdot 2}$$

$$a_4 = \frac{n(n+1)(n-2)(n+3)}{4!} a_0$$

Put $r = 3 \Rightarrow a_5 = \frac{a_3 [12 - n(n+1)]}{4\cdot 5}$

$$a_5 = \frac{-[n^2+n-12]}{4\cdot 5} a_3 = \frac{-[(n-3)(n+4)]}{4\cdot 5} a_3$$

$$a_5 = \frac{-\{(n-3)(n+4)\}}{4\cdot 5} \cdot \left\{ \frac{-(n-1)(n+2)}{2\cdot 3} \right\} a_1$$

$$a_5 = \frac{(n-1)(n+2)(n-3)(n+4)}{5!} a_1 \quad \text{and so on}$$

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = a_0 + a_1 x - x^2 \frac{n(n+1)}{2!} a_0 - \frac{(n-1)(n+2)}{3!} a_1 \cdot x^3$$

$$+ x^4 \frac{n(n+1)(n-2)(n+3)}{4!} a_0 + \frac{(n-1)(n+2)(n-3)(n+4)a_1}{5!} \cdot x^5 + - - - - -$$

$$y = a_0 \left\{ 1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^4 - \cdots \right\} +$$

$$a_1 \left\{ x - \frac{(n-1)(n+1)}{3!} x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!} x^5 - \cdots \right\} \quad \text{—— } (3)$$

$$y = a_0\, y_1(x) + a_1\, y_2(x) \quad \text{—— } (4)$$

where $y_1(x)$ and $y_2(x)$ represent two infinite series and Eqn (4) represents the series solution of Legendre's D.E

7c) Prove that the orthogonality property of Bessel's function as

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0 \quad \alpha \neq \beta$$

Sol? consider $J_n(\lambda x)$ is a solution of the Eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2) y = 0$$

Now if $u = J_n(\alpha x)$ $v = J_n(\beta x)$ then the diff^{al} Eqns are

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \quad \text{—— (1)}$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \quad \text{—— (2)}$$

Multiplying (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$

$$x v u'' + v u' + \alpha^2 u v x - \frac{n^2 u v}{x} = 0 \quad \text{—— (3)}$$

$$x u v'' + u v' + \beta^2 u v x - \frac{n^2 u v}{x} = 0 \quad \text{—— (4)}$$

eqn (3) − Eqn (4) ⟹

$$x[v u'' - u v''] + [v u' - u v'] + (\alpha^2 - \beta^2) u v x = 0$$

$$\frac{d}{dx}\left\{x[v u' - u v']\right\} = -(\alpha^2 - \beta^2) u v x$$

$$\frac{d}{dx}\left\{x[v u' - u v']\right\} = (\beta^2 - \alpha^2) u v x$$

Integrating bothsides w.r.t x between 0 to 1

$$\left\{x\left[vu' - uv'\right]\right\}_0^1 = (\beta^2 - \alpha^2)\int_0^1 x\, uv\, dx$$

$$\left[vu' - uv'\right]_{x=1} - 0 = (\beta^2 - \alpha^2)\int_0^1 x\, uv\, dx$$

$$\left[vu' - uv'\right]_{x=1} = (\beta^2 - \alpha^2)\int_0^1 x\, uv\, dx \quad \text{—— (5)}$$

since $u = J_n(\alpha x)$ $\quad v = J_n(\beta x)$

$$u' = J_n'(\alpha x)\cdot\alpha, \quad v' = J_n'(\beta x)\cdot\beta$$

$$\left[J_n(\beta x)\, J_n'(\alpha x)\cdot\alpha - J_n(\alpha x)\, J_n'(\beta x)\cdot\beta\right]_{x=1} =$$

$$(\beta^2 - \alpha^2)\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx.$$

$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx =$$

$$\frac{1}{(\beta^2 - \alpha^2)}\left[\alpha J_n(\beta x) J_n'(\alpha x) - \beta J_n(\alpha x) J_n'(\beta x)\right]_1 \quad \text{—— (6)}$$

since α and β are two distinct roots of $J_n(x) = 0$ $\Rightarrow J_n(\alpha) = 0$ $\quad J_n(\beta) = 0$

(6) $\Rightarrow$ since $\alpha \neq \beta$ ie $\beta^2 - \alpha^2 \neq 0$

$$\Rightarrow \int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = 0 \qquad \alpha \neq \beta.$$

8a) Test for convergence for $\Sigma\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Sol^n: Let $\Sigma u_n = \Sigma\left\{1+\frac{1}{\sqrt{n}}\right\}^{-n^{3/2}}$

$u_n = \left\{1+\frac{1}{\sqrt{n}}\right\}^{-n^{3/2}} \Rightarrow (u_n)^{1/n} = \left[\left\{1+\frac{1}{\sqrt{n}}\right\}^{-n^{3/2}}\right]^{\frac{1}{n}}$

$\{u_n\}^{1/n} = \left\{\frac{1}{\left\{1+\frac{1}{\sqrt{n}}\right\}^{n^{3/2}}}\right\}^{1/n}$

$\{u_n\}^{1/n} = \left\{\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}\right\}$

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left\{\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}\right\} = \frac{1}{e} < 1$

$\because \lim_{n\to\infty}\left\{1+\frac{1}{\sqrt{n}}\right\}^{\sqrt{n}} = e$

$\therefore$ By cauchy's Root test the series Σu_n is convergent.

8b) Find the series sol^n of Bessel differential Eqn $x^2 y'' + xy' + (x^2 - n^2)y = 0$ leading to $J_n(x)$

Sol: $x^2 y'' + xy' + (n^2 - x^2)y = 0$ ——— (1)

$P_0(x) = x^2 \Rightarrow P_0(x) = 0$ at $x = 0$

x is an singular point.

Now assume $y = \sum_{r=0}^{\infty} a_r x^{m+r}$ where $a_0 \neq 0$ ——— (2)

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}$$

$\therefore$ Eqn (1) becomes

$$x^2 \left\{ \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2} \right\} + x \left\{ \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1} \right\} + (x^2 - n^2) \left\{ \sum_{r=0}^{\infty} a_r x^{m+r} \right\} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r} + \sum_{r=0}^{\infty} a_r (m+r) x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\sum_{r=0}^{\infty} a_r x^{m+r}[(m+r)(m+r-1)+(m+r)-n^2]+$$

$$\sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r x^{m+r}[(m+r)(m+r\cancel{-1}\cancel{+1})-n^2]+$$

$$\sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r x^{m+r}[(m+r)^2-n^2]+\sum_{r=0}^{\infty} a_r x^{m+r+2} = 0 \quad \text{(3)}$$

Equating the coefficients of x^m [ie $r=0$ in (3)]

$a_0[m^2-n^2] = 0$ given that $a_0 \neq 0$

$\Rightarrow m^2-n^2 = 0 \Rightarrow m^2 = n^2 \Rightarrow m = \pm n$

$m = n$, $m = -n$

Equating the coefficients of x^{m+1} [ie $r=1$ in (3)]

$a_1[(m+1)^2-n^2] = 0 \Rightarrow a_1 = 0$

$\because (m+1)^2-n^2 \neq 0 \quad \because m = \pm n$

Equating the Coefficient of x^{m+r} in (3)

$a_r[(m+r)^2-n^2] + a_{r-2} = 0$

$$a_r\,[(m+r)^2 - n^2] = -a_{r-2}$$

$$a_r = \frac{-a_{r-2}}{(m+r)^2 - n^2}$$

At $m = n$ $\Rightarrow$ $a_r = -\dfrac{a_{r-2}}{(n+r)^2 - n^2}$

$$a_r = -\frac{a_{r-2}}{\cancel{n^2} + 2nr + r^2 - \cancel{n^2}} = \frac{-a_{r-2}}{2nr + r^2}$$

At $r = 2$ $\Rightarrow$ $a_2 = -\dfrac{a_0}{4n+4}$ $\Rightarrow$ $a_2 = \dfrac{-a_0}{4(n+1)}$

$r = 3 \Rightarrow a_3 = -\dfrac{a_1}{6n+9}$ $\Rightarrow a_3 = 0$ $\because a_1 = 0$

$r = 4 \Rightarrow a_4 = \dfrac{-a_2}{8n+16}$ $\Rightarrow a_4 = \dfrac{-1}{8n+16}\left\{\dfrac{-a_0}{4(n+1)}\right\}$

$$a_4 = \frac{a_0}{32(n+1)(n+2)}$$

From Eqn (2)

$$y = x^m\,[a_0 + a_1 x + a_2 x^2 + \cdots]$$

$$y_1 = x^n\,\left[a_0 + 0 + \left\{\frac{-a_0}{4(n+1)}\right\}x^2 + 0 + \left\{\frac{a_0}{32(n+1)(n+2)}\right\}x^4 + \cdots\right] \quad \because m = n$$

$$y_1 = a_0 x^n \left\{ 1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \cdots\cdots \right\} \quad \text{(4)}$$

At $m = -n$

$$y_2 = a_0 x^{-n} \left\{ 1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} - \cdots\cdots \right\} \quad \text{(5)}$$

The general solution of Bessel Eqn is

$$y = c_1 y_1 + c_2 y_2$$

Now choosing $a_0 = \dfrac{1}{2^n \Gamma n+1}$ in y_1

$$y_1 = \frac{x^n}{2^n \Gamma n+1} \left\{ 1 - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\cdot 2} - \cdots\cdots\cdots \right\}$$

$$y_1 = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\Gamma n+1} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma n+1} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\Gamma n+1 \cdot 2} - \cdots\cdots \right\}$$

w.k.t $\Gamma n = (n-1)\Gamma n-1$, $\Gamma n+1 = n \Gamma n$

$\Gamma n+2 = n+1 \; \Gamma n+1$

$\Gamma n+3 = n+2 \; \Gamma n+2 = (n+2)(n+1)\Gamma n+1$

$\Gamma n+3 = (n+1)(n+2)\Gamma n+1$ and so on

$$y_1 = \left(\frac{x}{2}\right)^n \left\{ \frac{(-1)^0}{\overline{|n+1}\ 0!} + \left(\frac{x}{2}\right)^2 \frac{(-1)^1}{\overline{|n+2}\ 1!} + \left(\frac{x}{2}\right)^4 \frac{(-1)^2}{\overline{|n+3} \cdot 2!} + \cdots\cdots \right\}$$

$$y_1 = \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\overline{|n+r+1} \cdot r!} \left(\frac{x}{2}\right)^{2r} \qquad \text{—— (6)}$$

Eqn (6) is called Bessel function of first kind of order r and is denoted by $J_n(x)$

$$\Rightarrow J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\overline{|n+r+1}\ r!} \left(\frac{x}{2}\right)^{n+2r}$$

8c) Express the polynomial x^3+2x^2-4x+5

Soln. $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$

$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$

$x = P_1(x)$

$$x^3+2x^2-4x+5 = \left\{\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right\}$$
$$+2\left\{\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right\} - 4P_1(x) + 5P_0(x)$$

$$x^3+2x^2-4x+5 = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{17}{5}P_1(x)$$
$$+\frac{17}{3}P_0(x)$$

Module - 5

9(a) Using Newton's forward difference formula find f(38)

x	40	50	60	70	80	90
f(x)	184	204	226	250	276	304

Sol^n. To find f(38) we use Newton's forward interpolation formula. The forward difference table is as follows.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
40	184			
		$\boxed{20}$		
50	204		$\boxed{2}$	
		22		0
60	226		2	
		24		0
70	250		2	
		26		0
80	276		2	
		28		
90	304			

$\Delta y_0 = 20$

$\Delta^2 y_0 = 2$

$$f(x) = y_0 + \frac{r}{1!}\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \cdots$$

$$r = \frac{x - x_0}{h} = \frac{38-40}{10} = \frac{-2}{10} = -0.2$$

$$f(38) = 184 + \frac{(-0.2)}{1}[20] + \frac{(-0.2)(-0.2-1)}{\cancel{2}}[\cancel{2}]$$

$f(38) = 184 - 4 + 0.24$

$f(38) = 180.24$

9(b) Find the real root of the equation $x \log_{10} x = 1.2$ by Regula - Falsi method between 2 and 3 (three iterations)

Sol?: $f(x) = x \log_{10} x - 1.2$

$f(2) = -0.5979 < 0$

$f(3) = 0.2314 > 0$

$f(2) \cdot f(3) < 0$

$\therefore$ A root lies between 2 and 3 ie (2, 3)

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$a = 2$

$f(a) = -0.5979$

$b = 3$

$f(b) = 0.2314$

$$x_1 = \frac{(2)(+0.2314) - (3)(-0.5979)}{0.2314 - (-0.5979)} = \frac{2.2565}{0.8293}$$

$x_1 = 2.72096 \simeq 2.7201$

$f(x_1) = -0.0179$

A root lies between $(2.7201, 3)$

$a = 2.7201$ $\quad$ $b = 3$

$f(a) = -0.0179$ $\quad$ $f(b) = 0.2314$

$$x_2 = \frac{(2.7201)(0.2314) - (3)(-0.0179)}{0.2314 - (-0.0179)}$$

$x_2 \simeq 2.7402.$

$f(x_2) = -3.8904 \times 10^{-4}$

A root lies between $(2.7402, 3)$

$a = 2.7402$ $\quad$ $b = 3$

$f(a) = -3.8904 \times 10^{-4}$ $\quad$ $f(b) = 0.2314$

$$x_3 = \frac{(2.7402)(0.2314) - (3)(-3.8904 \times 10^{-4})}{0.2314 - (-3.8904 \times 10^{-4})}$$

$x_3 = 2.7406$

After three iterations the required real root is 2.7406

9(c) Evaluate $\int_{4}^{5.2} \log x \, dx$ by weddle's rule considering six intervals.

Sol^n $I = \int_{4}^{5.2} \log x \, dx$, $h = \frac{b-a}{n} = \frac{5.2-4}{6} = \frac{1.2}{6}$

$h = 0.2$

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$y=f(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

weddle's rule for $n=6$ is given by

$$I = \frac{3}{10} h \left\{ y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 \right\}$$

$$I = \frac{3}{10}(0.2) \left\{ 1.3863 + 5(1.4351) + 1.4816 + 6(1.5261) + 1.5686 + 5(1.6094) + 1.6487 \right\}$$

$I = 1.827858$

$I \approx 1.8278$

10 a) Find f(9) from the data by Newton's divided difference formula

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Sol: Difference table is as follows.

x	$y=f(x)$	F.D.D $[x_0 x_1]$	S.D.D $[x_0 x_1 x_2]$	T.D.D $[x_0 x_1 x_2 x_3]$	F.D.D $[x_0 x_1 x_2 x_3 x_4]$
5	150				
		$\frac{392-150}{7-5}=121$			
7	392		$\frac{265-121}{11-5}=24$		
		$\frac{1452-392}{11-7}=265$		$\frac{32-24}{13-5}=1$	
11	1452		$\frac{437-265}{13-7}=32$		$\frac{1-1}{17-5}=0$
		$\frac{2366-1452}{13-11}=437$		$\frac{42-32}{17-7}=1$	
13	2366		$\frac{709-437}{17-11}=42$		
		$\frac{5202-2306}{17-13}=709$			
17	5202				

we have

$$y=f(x)=y_0+(x-x_0)[x_0 x_1]+(x-x_0)(x-x_1)[x_0 x_1 x_2] + \cdots$$

$$f(9)=150+(9-5)[121]+(9-5)(9-7)[24] + (9-5)(9-7)(9-11)[1]+0$$

$$f(9)=150+484+192-16$$

$$f(9)=810 \ //$$

10b) Using Newton Raphson method find the real root of the Equation $x \sin x + \cos x = 0$ near $x = \pi$

Sol: $f(x) = x \sin x + \cos x$

$f'(x) = x \cos x + \cancel{\sin x} - \cancel{\sin x} = x \cos x$

given that $x_0 = \pi$

Newton's Raphson iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, 3, \ldots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{f(\pi)}{f'(\pi)} = \pi - \frac{(-1)}{(-\pi)}$$

$$x_1 = \pi - \frac{1}{\pi} = 2.8233$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8233 - \frac{f(2.8233)}{f'(2.8233)}$$

$$x_2 = 2.8233 - \frac{(-0.06623)}{(-2.6815)} = 2.7986$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7986 - \frac{f(2.7986)}{f'(2.7986)}$$

$$x_3 = 2.7986 - \frac{(-5.6385 \times 10^{-4})}{(-2.6356)} = 2.7984$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.7984 - \frac{f(2.7984)}{f'(2.7984)}$$

$$x_4 = 2.7984 - \frac{(-3.6772 \times 10^{-5})}{(-2.6352)} = 2.7984$$

$$x_3 = x_4 = 2.7984$$

Hence required real root is 2.7484

10c) By using Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by considering 7 ordinates

Sol: Let $I = \int_0^6 \frac{dx}{1+x^2}$ $\quad h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
$y = 1/1+x^2$	1	0.5	0.2	0.1	0.0588	0.03846	0.02703
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule for $n=6$ is given by

$$I = \frac{h}{3}\left\{(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right\}$$

$$I = \frac{1}{3}\{(1+0.02703) + 2(0.2+0.0588) + 4(0.5+0.1+0.03846)\}$$

$$I = \frac{4.09847}{3} \simeq 1.366156667$$

$$I \simeq 1.3662$$