Second Semester B.E. Degree Examination, Aug./Sept.2020

Advanced Calculus and Numerical Methods

Time: 3 hrs. Max. Marks:100

Note: Answer any FIVE full questions, choosing ONE full question from each module.

Module-1

1 a. Find the angle between the surfaces $x^2 + y^2 - z^2 = 4$ and $z = x^2 + y^2 - 13$ at (2, 1, 2). **(06 Marks)**

b. If $\vec{F} = \nabla(xy^3z^2)$, find div $\vec{F}$ and curl $\vec{F}$ at (1,–1, 1). **(07 Marks)**

c. Find the value of the constant a such that the vector field

$$\vec{F} = (axy - z^3)\hat{i} + (a-2)x^2 j + (1-a)xz^2 k$$

is irrotational and hence find a scalar function ϕ such that $\vec{F} = \nabla\phi$. **(07 Marks)**

OR

2 a. If $\vec{F} = (3x^2 + 6y)i - 14yzj + 20xz^2k$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the curve given by $x = t$, $y = t^2$ and $z = t^3$. **(06 Marks)**

b. Use Green's theorem to find the area between the parabolas $x^2 = 4y$ and $y^2 = 4x$. **(07 Marks)**

c. If $\vec{F} = 2xyi + yz^2j + xzk$ and s is the rectangular parallelopiped bounded by x = 0, y = 0, z = 0 and x = 2, y = 1, z = 3. Find the flux across S. **(07 Marks)**

Module-2

3 a. Solve $(D^2 + 3D + 2)y = 4\cos^2 x$. **(06 Marks)**

b. Solve $(D^2 + 1)y = \sec x \tan x$, by the method of variation of parameter. **(07 Marks)**

c. Solve $x^2y'' + xy' + 9 = 3x^2 + \sin(3\log x)$. **(07 Marks)**

OR

4 a. Solve $y'' + 2y' + y = 2x + x^2$. **(06 Marks)**

b. Solve $(2x + 1)^2y'' - 6(2x + 1)y' + 16y = 8(2x + 1)^2$. **(07 Marks)**

c. The current i and the charge q in a series circuit containing on inductance L, capacitance C, emf E satisfy the differential equation : $L\frac{di}{dt} + \frac{q}{c} = E$; $i = \frac{dq}{dt}$. Express q and i interms of t, given that L, C, E are constants and the value of i, q are both zero initially. **(07 Marks)**

Module-3

5 a. Form the partial differential equation by eliminating the arbitrary function from $\phi(xy + z^2, x + y + z) = 0$. **(06 Marks)**

b. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when x = 0 and z = 0 if $y = (2n+1)\frac{\pi}{2}$. **(07 Marks)**

c. Derive one dimensional wave equation in the standard form $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$. **(07 Marks)**

OR

6 a. Form the partial differential equation by eliminating the arbitrary function form $f\left(\frac{xy}{z}, z\right) = 0$. **(06 Marks)**

b. Solve $\frac{\partial^2 z}{\partial y^2} = z$, given that when $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$. **(07 Marks)**

c. Find all possible solutions of one dimensional heat equation $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$ using the method of separation of variables. **(07 Marks)**

Module-4

7 a. Test for convergence of the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n, (x > 0)$. **(06 Marks)**

b. Solve the Bessel's differential equation leading to $J_n(x)$. **(07 Marks)**

c. Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ interms of Legendre's polynomials. **(07 Marks)**

OR

8 a. Test for convergence of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. **(06 Marks)**

b. If α and β are two distinct roots fo $J_n(x) = 0$. Prove that $\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = 0$. If $\alpha \neq \beta$. **(07 Marks)**

c. Express $f(x) = x^3 + 2x^2 - x - 3$ interms of Legendre's polynomials. **(07 Marks)**

Module-5

9 a. Find the real root of the equation : $x^3 - 2x - 5 = 0$ using Regula Falsi method, correct to three decimal places. **(06 Marks)**

b. Use Lagrange's formula, find the interpolating polynomial that approximates the function described by the following data :

x	0	1	2	5
f(x)	2	3	12	147

c. Evaluate $\int_0^1 \frac{x dx}{1 + x^2}$ by Weddle's rule, taking seven ordinates and hence find log e^2.

OR

10 a. Find the real root of the equation $xe^x - 2 = 0$ using Newton – Raphson method correct to three decimal places.

b. Use Newton's divided difference formula to find f(4) given the data :

x	0	2	3	6
f(x)	–4	2	14	158

c. Use Simpson's $\frac{3}{8}^{th}$ rule to evaluate $\int_1^4 e^{1/x} dx$.

1a) Find the angle between the surfaces $x^2+y^2-z^2=4$ and $z = x^2+y^2-13$ at $(2,1,2)$

Solⁿ The angle between the surfaces is defined to be equal to the angle betⁿ their normals and we know that $\nabla\phi$ is a vector normal to the surface.

we have the Eqn of the 2 surfaces given by

$$x^2+y^2-z^2=4 \quad \text{and} \quad x^2+y^2-z=13$$

$$\phi_1 = x^2+y^2-z^2 \qquad \phi_2 = x^2+y^2-z$$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\nabla\phi_1 = 2x i + 2y j - 2z k$$

$$(\nabla\phi_1)_{(2,1,2)} = 4i + 2j - 4k$$

$$\nabla\phi_2 = 2x i + 2y j - k$$

$$(\nabla\phi_2)_{(2,1,2)} = 4i + 2j - k$$

If θ is the angle betⁿ these normals we have

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$\cos\theta = \frac{(4)(4)+(2)(2)+(-4)(-1)}{\sqrt{16+4+16}\ \sqrt{16+4+1}}$$

$$= \frac{16+4+4}{\sqrt{36}\ \sqrt{21}}$$

$$\cos\theta = \frac{\cancel{24}^{\,4}}{\cancel{6}_{\,1}\sqrt{21}}$$

$$\cos\theta = \frac{4}{\sqrt{21}}$$

Thus $\theta = \cos^{-1}\dfrac{4}{\sqrt{21}}$ //

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1b) If $\vec{F} = \nabla(xy^3z^2)$, find div $\vec{F}$ and curl $\vec{F}$ at $(1,-1,1)$

Soln: Let $\phi = xy^3z^2$

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\vec{F} = y^3z^2 i + 3xy^2z^2 j + 2xy^3z\, k$$

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial \vec{F}}{\partial x} i + \frac{\partial \vec{F}}{\partial y} j + \frac{\partial \vec{F}}{\partial z} k$$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xy^3z)$$

$$= 0 + 6xyz^2 + 2xy^3$$

$$\text{div}\,\vec{F} = 2xy\,[3z^2 + y^2]$$

$$\text{div}\,\vec{F}_{(1,-1,1)} = 2(1)(-1)\left[3(1)^2 + (-1)^2\right]$$

$$\text{div}\,\vec{F}_{(1,-1,1)} = -2\cdot[3+1] = -8 \;// \quad \boxed{\text{div}\,\vec{F} = -8}$$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$\text{curl}\,\vec{F} = i[\cancel{6xy^2z} - \cancel{6xy^2z}] - j[\cancel{2y^3z} - \cancel{2y^3z}]$$
$$+ k[\cancel{3y^2z^2} - \cancel{3y^2z^2}] = 0 \quad \boxed{\text{curl}\,\vec{F} = 0}$$

1c) Find the value of the constant 'a' such that the vector field

$\vec{F} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$

Sol: To find 'a' such that $\text{Curl}\,\vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$$i\{0 - 0\} - j\{(1-a)z^2 - (-3z^2)\}$$

$$+ k\{2x(a-2) - ax\} = 0$$

$$(a-4)z^2 j + (a-4)xk = 0$$

The above eqn identically satisfied when $a = 4$

Now consider, $\nabla\phi = (\vec{F})_{a=4}$

$$\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = (4xy - z^3)i + 2x^2 j - 3xz^2 k$$

$$\frac{\partial\phi}{\partial x} = 4xy - z^3 \quad \frac{\partial\phi}{\partial y} = 2x^2 \quad \frac{\partial\phi}{\partial z} = -3xz^2$$

consider $\frac{\partial\phi}{\partial x} = 4xy - z^3$

$$\phi = \int (4xy - z^3)\,dx + f_1(y,z)$$

$$\phi = 2x^2 y - xz^3 + f_1(y,z) \quad \text{———} (1)$$

consider $\frac{\partial\phi}{\partial y} = 2x^2$

$\phi = \int 2x^2\,dy + f_2(x,z) = 2x^2y + f_2(x,z)$ — (2)

consider $\frac{\partial\phi}{\partial z} = -3xz^2$

$\phi = -\int 3xz^2\,dz + f_3(x,y) = -xz^3 + f_3(x,y)$ — (3)

From Eqn (1) (2) and (3) we get

$f_1(y,z) = 0$, $f_2(x,z) = -xz^3$ $f_3(x,y) = 2x^2y$

Thus $\phi = 2x^2y - xz^3$ //

Prof Vijaya.c

2a) If $\vec{F} = (3x^2+6y)i - 14yzj + 20xz^2k$

Evaluate $\int_C \vec{F}\,\vec{dr}$ from $(0,0,0)$ to $(1,1,1)$

along the curve given by $x=t$ $y=t^2$

$z=t^3$

Soln. $\vec{F} = (3x^2+6y)i - 14yzj + 20xz^2k$

$\vec{r} = xi + yj + zk$ $\quad d\vec{r} = dxi + dyj + dzk$

$\vec{F}\cdot\vec{dr} = (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz$

Put $x=t$, $y=t^2$, $z=t^3$

$dx = dt$ $\quad dy = 2t\,dt$ $\quad dz = 3t^2dt$

$\vec{F}\cdot\vec{dr} = (3t^2+6t^2)dt - (14t^5)(2t\,dt) + 20t^7(3t^2)dt$

$\vec{F}\cdot\vec{dr} = (9t^2 - 28t^6 + 60t^9)\,dt$ $\quad 0 \le t \le 1$

$$\int_C \vec{F}\cdot\vec{dr} = \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left\{ \cancel{9}^{3}\frac{t^3}{\cancel{3}} - \cancel{28}^{4}\frac{t^7}{\cancel{7}} + \cancel{60}^{6}\frac{t^{10}}{\cancel{10}} \right\}_0^1$$

$$\int_C \vec{F}\cdot\vec{dr} = \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1 = \left[(3-4+6) - 0\right]$$

$$\int_C \vec{F}\,\vec{dr} = 5 \quad //$$

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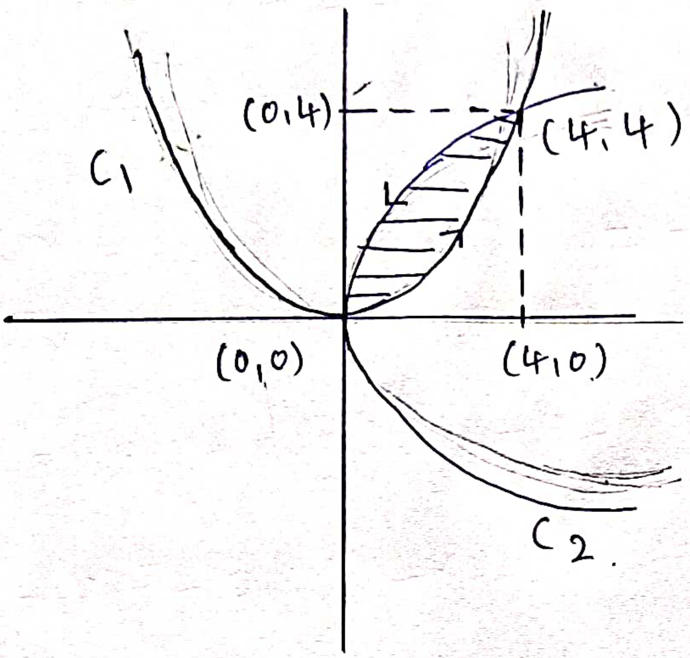
2b) Use Green's theorem to find the area between parabolas $x^2 = 4y$ $y^2 = 4x$

Sol: we have the area

$$A = \iint dx\,dy = \frac{1}{2}\int_C x\,dx - y\,dy$$

let us find the point of intersection of $y^2 = 4x$ and $x^2 = 4y$



$$x^2 = 4y \Rightarrow y = \frac{x^2}{4}$$

consider $y^2 = 4x \Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x$

$$x^4 = 64x \Rightarrow x^4 - 64x = 0 \Rightarrow x(x^3 - 64) = 0$$

$$x = 0, \quad x = 4 \Rightarrow y = 0, \quad y = 4$$

Thus $(0,0)$ and $(4,4)$ are point of intersection.

C_1 is the curve $x^2 = 4y \Rightarrow \cancel{2}x\,dx = \cancel{4}^{2}dy$

$0 \le x \le 4$ $\quad dy = \frac{x}{2}dx$

C_2 is the curve $y^2 = 4x \Rightarrow \cancel{2}y\,dy = \cancel{4}^{2}dx$

$4 \le y \le 0$ $\quad dx = \frac{y}{2}dy$

$$A = \frac{1}{2}\int_{C_1} x\,dy - y\,dx + \frac{1}{2}\int_{C_2} x\,dy - y\,dx$$

$$A = \frac{1}{2}\int_{x=0}^{4} x \cdot \frac{x}{2}dx - \frac{x^2}{4}dx \; +$$

$$\frac{1}{2}\int_{y=4}^{0} \frac{y^2}{4}dy - y \cdot \frac{y}{2}dy$$

$$A = \frac{1}{2}\int_{x=0}^{4}\left(\frac{x^2}{2} - \frac{x^2}{4}\right)dx + \frac{1}{2}\int_{y=4}^{0}\left(\frac{y^2}{4} - \frac{y^2}{2}\right)dy$$

$$A = \frac{1}{2}\int_{x=0}^{4}\frac{x^2}{4}dx + \frac{1}{2}\int_{y=4}^{0} -\frac{y^2}{4}dy$$

$$A = \frac{1}{2}\int_{x=0}^{4}\frac{x^2}{4}dx - \frac{1}{2}\int_{y=4}^{0}\frac{y^2}{4}dy$$

$$= \frac{1}{8}\left[\frac{x^3}{3}\right]_0^4 - \frac{1}{8}\left[\frac{y^3}{3}\right]_4^0$$

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$$A = \frac{1}{24}\left\{4^3 - 0\right\} - \frac{1}{24}\left\{0 - 4^3\right\}$$

$$= \frac{64}{24} - \frac{(-64)}{24}$$

$$= \frac{64}{24} + \frac{64}{24} = \frac{\cancel{128}^{16}}{\cancel{24}_{3}}$$

$$A = \frac{16}{3}$$

Thus the required area is $\frac{16}{3}$ Sq. units.

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2c) If $\vec{F} = 2xy\hat{i} + yz^2\hat{j} + xzk$ and S is the rectangular parallelopiped bounded by $x=0$, $y=0$, $z=0$ and $x=2$, $y=1$ $z=3$. Find the flux across S

Soln. Flux across $S = \iint_S \vec{F}\cdot\hat{n}\,ds$

By divergence theorem $\iint_S \vec{F}\cdot ds = \iiint_V \operatorname{div}\vec{F}\,dV$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}\right)\vec{F}$$

$$\operatorname{div}\vec{F} = 2y + z^2 + x$$

$$\iint_S \vec{F}\,\hat{n}\,ds = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3}(2y + z^2 + x)\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{1}\left[2yz + \frac{z^3}{3} + xz\right]_0^3 dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{1}\left[6y + \frac{\cancel{27}^{\,9}}{\cancel{3}} + 3x - (0)\right]dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{1}\left[6y + 3x + 9\right]dy\,dx$$

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$$\iint_S \vec{F}\,\hat{n}\,ds = \int_{x=0}^{2} \left[\cancel{6}^{3}\frac{y^2}{\cancel{2}_1} + 3xy + 9y\right]_0^1 dx$$

$$= \int_{x=0}^{2} [3 + 3x + 9]\,dx = \int_{x=0}^{2}[3x+12]\,dx$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \left[3\frac{x^2}{2} + 12x\right]_0^2$$

$$= \frac{3}{2}[4-0] + 12[2-0]$$

$$= \frac{3}{\cancel{2}}\cdot\cancel{4}^{2} + 24 = 6+24 = 30$$

$$\iint_S \vec{F}\,\hat{n}\,ds = 30$$

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3a) Solve $(D^2+3D+2)y = 4\cos^2 x$

Sol: A.E is $m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0$

$m=-1,-2$

$y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$

$4\cos^2 x = \cancel{4}^{2}\left\{\frac{1+\cos 2x}{\cancel{2}}\right\} = 2(1+\cos 2x)$

$4\cos^2 x = 2 + 2\cos 2x$

$$y_p(x) = \frac{2+2\cos 2x}{D^2+3D+2}$$

$$= \frac{2}{D^2+3D+2} + \frac{2\cos 2x}{D^2+3D+2}$$

$y_p(x) = y_{p_1}(x) + y_{p_2}(x)$

$$y_{p_1}(x) = \frac{2}{D^2+3D+2} = \frac{2\cdot e^{0\cdot x}}{D^2+3D+2} = \frac{2\cdot e^{0\cdot x}}{0+0+2}$$

$\because D=a=0$

$$y_{p_1}(x) = \frac{\cancel{2}}{\cancel{2}} = 1$$

$$y_{p_2}(x) = \frac{2\cos 2x}{D^2+3D+2}$$ Put $D^2=-2^2=-4$ in $f(D)$

$$y_{p_2}(x) = \frac{2\cos 2x}{-4+3D+2} = \frac{2\cos 2x}{3D-2}$$

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$$y_{p_2}(x) = \frac{2\cos 2x}{3D-2} \times \frac{3D+2}{3D+2}$$

$$y_{p_2}(x) = \frac{2[3D\cos 2x + 2\cos 2x]}{(3D)^2 - (2)^2}$$

$$y_{p_2}(x) = \frac{6(-\sin 2x)\cdot 2 + 2\cos 2x}{9D^2 - 4}$$ Put $D^2 = -4$ in DR

$$y_{p_2}(x) = \frac{-12\sin 2x + 4\cos 2x}{9(-4) - 4}$$

$$= \frac{+\cancel{4}[3\sin 2x - \cos 2x]}{+\cancel{40}_{10}}$$

$$y_{p_2}(x) = \frac{1}{10}(3\sin 2x - \cos 2x)$$

$$y_p(x) = 1 + \frac{1}{10}(3\sin 2x - \cos 2x)$$

General solution : $y = y_c(x) + y_p(x)$

$$y = c_1 e^{-x} + c_2 e^{-2x} + 1 + \frac{1}{10}(3\sin 2x - \cos 2x)$$

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3b) Solve $(D^2+1)y = \sec x \tan x$, by method of variation of parameter.

Sol: A.E is $m^2+1=0 \Rightarrow m^2=-1=i^2 \Rightarrow m=\pm i$

$y_c(x) = c_1\cos x + c_2 \sin x \Rightarrow y_1(x)=\cos x \quad y_2(x)=\sin x$

Let us assume that $y_p(x) = u_1(x)\,y_1(x)+u_2(x)\,y_2(x)$

$u_1(x) = -\int \frac{X\,y_2(x)}{W}dx \qquad u_2(x) = \int \frac{X\,y_1(x)}{W}dx$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x)$$

$W = \cos^2 x + \sin^2 x = 1 \qquad \boxed{W=1}$

$$u_1(x) = -\int \frac{\sec x \tan x}{1}\sin x\,dx$$

$$u_1(x) = -\int \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\cdot \sin x\,dx$$

$$= -\int \frac{\sin^2 x}{\cos^2 x}dx = -\int \tan^2 x\,dx$$

$$= -\int (\sec^2 x - 1)\,dx = \int (1-\sec^2 x)\,dx$$

$u_1(x) = x - \tan x$

$$u_2(x) = \int \frac{\sec x \tan x}{1}\cdot \cos x\,dx$$

$$u_2(x) = \int \frac{1}{\cancel{\cos x}} \cdot \tan x \, \cancel{\cos x} \, dx$$

$$u_2(x) = \int \tan x \, dx = \log \sec x$$

$$y_p(x) = (x - \tan x)\cos x + (\log \sec x)\sin x$$

General Solution (G.S) = C.F + P.I

$$y = y_c(x) + y_p(x)$$

$$y = y_1 \cos x + y_2 \sin x + (x - \tan x)\cos x + (\log \sec x)\sin x$$

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3c) Solve $x^2y'' + xy' + 9y = 3x^2 + \sin(3\log x)$

Sol^n. Put $x = e^t \Rightarrow t = \log x$.

$xy' = Dy \Rightarrow x^2y'' = D(D-1)y \qquad D = \frac{d}{dt}$

The given Eqn becomes

$[D(D-1)y + Dy + 9y] = 3(e^t)^2 + \sin 3t$

$[D^2 - \cancel{D} + \cancel{D} + 9]y = 3e^{2t} + \sin 3t$

$(D^2+9)y = 3e^{2t} + \sin 3t$

A.E is $m^2 + 9 = 0 \Rightarrow m^2 = -9 = 9i^2$

$\Rightarrow m = \pm i3$

$y_c = c_1 \cos 3t + c_2 \sin 3t$

$$y_p = \frac{3e^{2t} + \sin 3t}{D^2+9} = \frac{3e^{2t}}{D^2+9} + \frac{\sin 3t}{D^2+9}$$

$y_p = y_{p_1} + y_{p_2}$

$$y_{p_1} = \frac{3e^{2t}}{D^2+9} \qquad \text{Put } D = 2 \text{ in } f(D)$$

$$y_{p_1} = 3\frac{e^{2t}}{4+9} = \frac{3}{13}e^{2t}$$

$$y_{p_2} = \frac{\sin 3t}{D^2+9} \qquad \text{Put } D^2 = -3^2 = -9 \text{ in } f(D)$$

$$y_{P_2} = \frac{\sin 3t}{-9+9} = \frac{\sin 3t}{0}$$

$$y_{P_2} = t \cdot \frac{\sin 3t}{f'(D)} = t\,\frac{\sin 3t}{2D} = \frac{t}{2}\,\frac{1}{D}\sin 3t$$

$$y_{P_2} = \frac{t}{2}\,\frac{(-\cos 3t)}{3} = -\frac{t\cos 3t}{6}$$

$$y_p = \frac{3}{13}e^{2t} - \frac{t\cos 3t}{6}$$

G.S = CF + P.I , $y = y_c + y_p$

$$y = c_1\cos 3t + c_2\sin 3t + \frac{3}{13}e^{2t} - \frac{t\cos 3t}{6}$$

Put $e^t = x \Rightarrow e^{2t} = x^2$

$t = \log x$

$$\Rightarrow y(x) = c_1\cos 3\log x + c_2\sin 3\log x + \frac{3}{13}x^2 - \frac{\log x\cos 3\log x}{6}$$

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4a) Solve $y'' + 2y' + y = 2x + x^2$

Sol: Put $y' = Dy \Rightarrow y'' = D^2 y$

$$\Rightarrow D^2 y + 2Dy + y = 2x + x^2$$

$$(D^2 + 2D + 1)\, y = 2x + x^2$$

A E is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow \cancel{m+1=0}$

$m = -1, -1$

$$y_c(x) = (C_1 + x C_2)\, e^{-x}$$

$$y_p(x) = \frac{x^2 + 2x}{D^2 + 2D + 1} = \frac{x^2 + 2x}{1 + 2D + D^2}$$

$$1 + 2D + D^2 \,\big)\, x^2 + 2x \,\big(\, x^2 - 2x + 2$$

$$\underline{\cancel{x^2} + 4x + 2}$$

$$-2x - 2$$

$$\underline{-\cancel{2x} - 4}$$

$$\cancel{2} + 0 + 0$$

$$\underline{\cancel{2}}$$

$$0$$

$$\boxed{y_p(x) = x^2 - 2x + 2}$$

Thus $y = y_c + y_p$

$$y = (C_1 + x C_2)\, e^{-x} + (x^2 - 2x + 2)$$

Prof Vijaya.C

4b) Solve $(2x+1)^2 y'' - 6(2x+1)y' + 16y = 8(2x+1)^2$

Sol: Put $2x+1 = e^t \Rightarrow t = \log(2x+1)$

$D = \frac{d}{dt}$, $(2x+1)y' = 2Dy$

$(2x+1)^2 y'' = 2^2 D(D-1)y$

Hence given Eqn becomes

$2^2 D(D-1)y - 6 \cdot 2 Dy + 16y = 8 \cdot (e^t)^2$

$4(D^2 - D)y - 12Dy + 16y = 8e^{2t}$

$\cancel{4}\{D^2 - D - 3D + 4\}y = \overset{2}{\cancel{8}} e^{2t}$

$\{D^2 - 4D + 4\}y = 2e^{2t}$

A.E is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$

$m = 2, 2$

$y_c(t) = (C_1 + t\,C_2)e^{2t}$

$y_p(t) = \frac{2e^{2t}}{D^2 - 4D + 4}$ Put $D = 2$ in $f(D)$

$y_p(t) = 2 \frac{e^{2t}}{2^2 - 4(2) + 4} = \frac{2e^{2t}}{8-8} = \frac{2e^{2t}}{0}$

$y_p(t) = 2 \cdot t \frac{e^{2t}}{f'(D)} = 2t \cdot \frac{e^{2t}}{2D-4}$ Put $D=2$ in $f'(D)$

$= 2t \frac{e^{2t}}{2(2)-4} = 2t \cdot \frac{e^{2t}}{0}$

$$y_p(t) = 2t^2 \frac{e^{2t}}{f'(D)} = \cancel{2}t^2 \frac{e^{2t}}{\cancel{2}}.$$

$$y_p(t) = t^2 e^{2t}$$

$$G.S = y_c + y_p.$$

$$y = (C_1 + tC_2)\, e^{2t} + t^2 e^{2t}$$

put $e^t = (2x+1) \Rightarrow e^{2t} = (2x+1)^2.$

$$t = \log(2x+1)$$

$$\Rightarrow y(x) = [C_1 + C_2 \log(2x+1)] + [\log(2x+1)]^2 \cdot (2x+1)^2$$

4c) The current i and the charge q in a series circuit containing on inductance L capacitance C, emf E satisfy the D.E $L\frac{di}{dt} + \frac{q}{C} = E$, $i = \frac{dq}{dt}$, Express q and i interms of t. i and q are both zero initially.

soln. $i = \frac{dq}{dt}$ using in the given D.E

$$L\frac{d}{dt}\left(\frac{dq}{dt}\right) + \frac{q}{C} = E \Rightarrow L\frac{d^2q}{dt^2} + \frac{q}{C} = E.$$

$$\frac{d^2q}{dt^2} + \frac{q}{CL} = \frac{E}{L}.$$

Denoting $\lambda^2 = \frac{1}{LC}$ and $\mu = \frac{E}{L}$ we have

$$(D^2+\lambda^2)q = \mu$$

A E is $m^2+\lambda^2 = 0 \Rightarrow m^2 = -\lambda^2 = i^2\lambda^2$

$m = \pm i\lambda$

$q_c = C.F = c_1\cos\lambda t + c_2\sin\lambda t$

$q_p = P.I = \frac{\mu}{D^2+\lambda^2} = \frac{\mu . e^{0.t}}{D^2+\lambda^2}$ Put $D = 0$ in $f(D)$

$$q_p = \frac{\mu e^{0 \cdot t}}{0+\lambda^2} = \frac{\mu}{\lambda^2}$$

$$q = q_c + q_p$$

$$q(t) = c_1 \cos\lambda t + c_2 \sin\lambda t + \frac{\mu}{\lambda^2} \quad \text{—— (1)}$$

$$q'(t) = -\lambda c_1 \sin\lambda t + \lambda c_2 \cos\lambda t \quad \text{—— (2)}$$

But $q(0) = 0$ $\quad q'(0) = 0$ by data

Hence Eqn (1) and Eqn (2) becomes

$$q(0) = c_1 + 0 + \frac{\mu}{\lambda^2} \Rightarrow 0 = c_1 + \frac{\mu}{\lambda^2}$$

$$c_1 = -\frac{\mu}{\lambda^2}.$$

$$q'(0) = 0 + \lambda c_2 \Rightarrow 0 = \lambda c_2 \Rightarrow c_2 = 0$$

substituting in Eqn (1)

$$q(t) = -\frac{\mu}{\lambda^2} \cos\lambda t + 0 + \frac{\mu}{\lambda^2}$$

$$q(t) = \frac{\mu}{\lambda^2}\left[1 - \cos\lambda t\right]$$

$$\frac{\mu}{\lambda^2} = \frac{E/L}{1/LC} = EC \quad \text{Thus}$$

$$q(t) = EC\left[1 - \cos\sqrt{1/LC}\cdot t\right] \quad //$$

$$i(t) = q'(t) = E\sqrt{C/L}\cdot \sin\sqrt{1/LC}\cdot t \quad //$$

5a) Form the PDE by eliminating arbitrary fun $\phi(xy+z^2, x+y+z)=0$

Sol: we have $\phi(xy+z^2, x+y+z)=0$ ——— (1)

$\phi(u,v)=0$ where $u=xy+z^2$ $v=x+y+z$

$\frac{\partial u}{\partial x} = y+2zp$ $\frac{\partial v}{\partial x} = 1+p$

$\frac{\partial u}{\partial y} = x+2zq$ $\frac{\partial v}{\partial y} = 1+q$

Diffate Eqn (1) partially w.r.t x and w.r.t y

$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x}$ ——— (2)

$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y}$ ——— (3)

Dividing Eqn (2) by Eqn (3)

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} \Rightarrow \frac{y+2zp}{x+2zq} = \frac{1+p}{1+q}$$

$(y+2zp)(1+q) = (x+2zq)(1+p)$

$y+yq+2zp+2zpq = x+xp+2zq+2zpq$

$y-x+yq-2zq+2zp-px+\cancel{2zpq}-\cancel{2zpq}=0$

$(y-x)+q(y-2z)+p(2z-x)=0$

$p(2z-x)+q(y-2z)+(y-x)=0$ is required P.D.E

5b) Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$ and $z = 0$ if $y = (2n+1)\pi/2$

Soln: $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ Integrating w.r.t x treating y as constant

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \quad \text{——(1)}$$

Integrating w.r.t y treating x as constant

$$z = -\cos x(-\cos y) + \int f(y)\,dy + g(x)$$

$$z = \cos x \cos y + F(y) + g(x) \quad \text{——(2)}$$

where $F(y) = \int f(y)\,dy$

Using the data $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$ in Eqn (1)

$-2\sin y = -\sin y + f(y) \Rightarrow f(y) = -\sin y$

$F(y) = \int f(y)\,dy \Rightarrow F(y) = \int(-\sin y)\,dy$

$F(y) = \cos y$

Using the data $z = 0$ if $y = (2n+1)\frac{\pi}{2}$ in Eqn (2)

$0 = \cos x \cos(2n+1)\frac{\pi}{2} + \cos(2n+1)\frac{\pi}{2} + g(x)$

But $\cos(2n+1)\frac{\pi}{2} = 0 \Rightarrow 0 = 0 + 0 + g(x)$

$g(x) = 0$. Thus soln of PDE is

$z = \cos x \cos y + \cos y$ //

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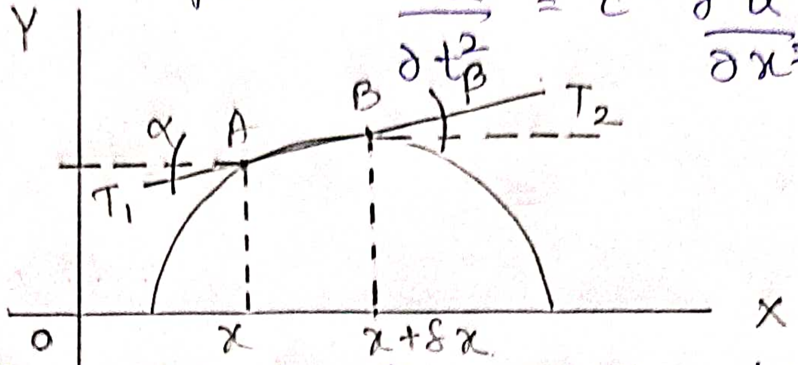
5c) Derive one dimemensional wave eqn in the standard form $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Sol:



consider the flexible string tightly stretched betn two pts at a distance l apart. Let ρ be mass per unit length of the string we shall assume that

i) Tension T of the string is same throughout

ii) The effect of gravity can be ignored due to large tension T

iii) the motion of the string is in small transverse vibrations.

Let us consider the forces acting on a small element AB of length δx. Let T_1 and T_2 be the tensions at the points A and B. since there is no motion in horizontal components T_1 & T_2 must cancel each other

$$\therefore T_1 \cos\alpha = T_2 \cos\beta = T \quad \text{---(1)}$$

where α and β are the angles made by T_1 and T_2 with the horizontal.

Vertical components of tension are $-T_1 \sin\alpha$ and $T_2 \sin\beta$, where negative sign is used because T_1 is directed downwards.

Hence resultant force acting vertically upwards is $T_2 \sin\beta - T_1 \sin\alpha$.

Applying Newton's second law of motion

Force = mass × Accelaration

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\, \delta x \frac{\partial^2 u}{\partial t^2}$$

Dividing throughout by T, we have

$$\frac{T_2}{T} \sin\beta - \frac{T_1}{T} \sin\alpha = \frac{\rho}{T}\, \delta x \frac{\partial^2 u}{\partial t^2}.$$

From Eqn (1) $\frac{T_1}{T} = \frac{1}{\cos\alpha}$ $\quad \frac{T_2}{T} = \frac{1}{\cos\beta}$

$$\frac{\sin\beta}{\cos\beta} - \frac{\sin\alpha}{\cos\alpha} = \frac{\rho}{T}\, \delta x \frac{\partial^2 u}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{\rho}{T}\, \delta x \frac{\partial^2 u}{\partial t^2} \quad \text{——(2)}$$

$\tan\alpha$ and $\tan\beta$ represents slopes at $A(x)$ and $B(x+\delta x)$ respeetively

$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x$ $\qquad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

Eqn (2) becomes

$$\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{\rho}{T}\, \delta x \frac{\partial^2 u}{\partial t^2}.$$

$$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

Taking limit as $\delta x \to 0$ we have

$$\lim_{\delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

LHS is nothing but the derivative of $\frac{\partial u}{\partial x}$ w.r.t x treating t as constant ie

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2}.$$

$u_{tt} = c^2 u_{xx}$ where $c^2 = \frac{T}{\rho}$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ which is required wave Eqn

6a) Form the PDE by eliminating the arbitrary function from $f\left(\frac{xy}{z}, z\right) = 0$

Soln: $f\left(\frac{xy}{z}, z\right) = 0$

$f(u, v) = 0$ where $u = \frac{xy}{z}$ $v = z$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \quad -(1)$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \quad -(2)$$

Eqn (1) ÷ Eqn (2)

$$\frac{\cancel{\frac{\partial f}{\partial u}}\frac{\partial u}{\partial x}}{\cancel{\frac{\partial f}{\partial u}}\frac{\partial u}{\partial y}} = \frac{+\cancel{\frac{\partial f}{\partial v}}\frac{\partial v}{\partial x}}{+\cancel{\frac{\partial f}{\partial v}}\frac{\partial v}{\partial y}} \Rightarrow \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

$$\frac{\partial u}{\partial x} = y\left\{\frac{z \cdot 1 - x \cdot \frac{\partial z}{\partial x}}{z^2}\right\} = y\left\{\frac{z - xp}{z^2}\right\}$$

$$\frac{\partial u}{\partial y} = x\left\{\frac{z \cdot 1 - y\frac{\partial z}{\partial y}}{z^2}\right\} = x\left\{\frac{z - yq}{z^2}\right\}$$

$$\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = p$$

$$\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = q$$

$$\Rightarrow \frac{\frac{yz - xyp}{\cancel{z^2}}}{\frac{xz - xyq}{\cancel{z^2}}} = \frac{p}{q}$$

$$q(yz - xyp) = p(xz - xyq)$$

$$qyz - xypq = pxz - xypq$$

$$pxz - qyz - \cancel{xypq} + \cancel{xypq} = 0$$

$$pxz - qyz = 0$$

$$\Rightarrow z.(px - qy) = 0$$

$$\Rightarrow px - qy = 0.$$ is required PDE

6b) solve $\frac{\partial^2 z}{\partial y^2} = z$ given that when $y=0$

$z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$

sol^n. suppose that z is a function of y only

The PDE assumes the form of ODE

$\frac{d^2 z}{dy^2} = z \Rightarrow D^2 z = z$ where $D = \frac{d}{dy} \Rightarrow D^2 = \frac{d^2}{dy^2}$

$D^2 z - z = 0 \Rightarrow (D^2 - 1) z = 0$

A.E is $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

Soln of ODE is $z = c_1 e^y + c_2 e^{-y}$

Soln of PDE is $z = f(x) e^y + g(x) e^{-y}$ —— (1)

Diff^ate Eqn (1) partially w.r.t y

$\frac{\partial z}{\partial y} = f(x) e^y + g(x) e^{-y} (-1)$

$\frac{\partial z}{\partial y} = f(x) e^y - g(x) e^{-y}$ —— (2)

using conditions $z = e^x$ when $y = 0$ in Eqn (1)

$e^x = f(x)(1) + g(x)(1)$

$e^x = f(x) + g(x)$ —— (3)

using conditions $\frac{\partial z}{\partial y} = e^{-x}$ when $y = 0$

$e^{-x} = f(x) - g(x)$ ——— (4)

solving Eqn (3) and Eqn (4)

$$f(x) + \cancel{g(x)} = e^{x}$$
$$f(x) - \cancel{g(x)} = e^{-x}$$

$$2f(x) = e^{x} + e^{-x}$$

$$f(x) = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

$$\Rightarrow g(x) = \frac{e^{x} - e^{-x}}{2} = \sinh x$$

Substituting in Eqn (1)

$$z = \cosh x \, e^{y} + \sinh x \, e^{-y}$$

//

6C) Find all possible solutions of one dimensional heat Eqn $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ using the method of separation of variables.

Soln consider $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ ——(1)

let $u = XT$ where $X = X(x)$ $T = T(t)$ be the solution of PDE

Hence Eqn (1) becomes

$$\frac{\partial}{\partial t}(XT) = c^2 \frac{\partial^2}{\partial x^2}(XT)$$

$$X \frac{dT}{dt} = Tc^2 \frac{d^2X}{dx^2} \Rightarrow \frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

Equating bothsides to a common constant k we have

$$\frac{1}{c^2T}\frac{dT}{dt} = k \qquad \frac{1}{X}\frac{d^2X}{dx^2} = k$$

$$\frac{dT}{dt} = kc^2T \qquad \frac{d^2X}{dx^2} = kX$$

$$(D - kc^2)T = 0 \quad \text{——(2)} \qquad (D^2 - k)X = 0 \quad \text{——(3)}$$

where $D = \frac{d}{dt}$

where $D = \frac{d}{dx} \Rightarrow D^2 = \frac{d^2}{dx^2}$

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case(i) when $k = 0$

Eqn (3) and Eqn (2) becomes

$D^2 X = 0 \Rightarrow \frac{d^2X}{dx^2} = 0$ Intigrating w.r.t x two times

$\frac{dX}{dx} = C_1$. Again integrating w.r.t x

$X = C_1 x + C_2$

$DT = 0 \Rightarrow \frac{dT}{dt} = 0$ Integrating w.r.t t

$T = C_3$

Hence solution of PDE is given by $u = XT$

$u = (C_1 x + C_2) C_3$

case(ii) let k is positive ie $k = +p^2$

Eqn (3) and Eqn (2) becomes

$(D^2 - p^2) X = 0$ A.E is $m^2 - p^2 = 0 \Rightarrow m^2 = p^2$

$m = \pm p$

$X = c_1' e^{px} + c_2' e^{-px}$

$(D - p^2c^2) T = 0 \Rightarrow DT = p^2c^2T \Rightarrow \frac{dT}{dt} = p^2c^2T$

$\frac{1}{T} dT = p^2c^2 dt$ Integrating

$\log T = p^2c^2t + k \Rightarrow T = e^{p^2c^2t} \cdot e^k$

$T = c_3' e^{p^2c^2t}$ where $c_3' = e^k$

Hence soln of PDE when $k = p^2$ is given by $u = XT$

$u = [c_1' e^{px} + c_2' e^{-px}] \cdot c_3' e^{c^2p^2t}$ //

Case (iii) when k is negative ie $k = -p^2$

Eqn (3) and eqn (2) becomes

$[D^2 - (-p^2)]X = 0 \Rightarrow [D^2 + p^2]X = 0$

A.E is $m^2 + p^2 = 0 \Rightarrow m^2 = -p^2 \Rightarrow m^2 = i^2 p^2$

$m = \pm ip$

$X = c_1'' \cos px + c_2'' \sin px$

$[D - (-p^2)c^2]T = 0 \Rightarrow [D + p^2c^2]T = 0$

$DT = -p^2c^2T \Rightarrow \frac{dT}{dt} = -p^2c^2T \Rightarrow \frac{1}{T}dT = -p^2c^2 dt$

Integrating $\log T = -p^2c^2t + k$

$T = e^{-p^2c^2t} \cdot e^k = c_3'' e^{-p^2c^2t}$

Hence soln of PDE when $k = -p^2$ is given by

$u = XT$ ie

$u = [c_1'' \cos px + c_2'' \sin px] \cdot c_3'' e^{-p^2c^2t}$

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7a) Test for convergence of series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ $(x>0)$

Sol^n. By data

$$u_n = \frac{(n!)^2}{(2n)!} x^n \Rightarrow u_{n+1} = \frac{[(n+1)!]^2}{[2(n+1)]!} x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{[(n+1)!]^2 x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2 x^n}$$

$$= \left\{\frac{(n+1)!}{n!}\right\}^2 \cancel{x^n} \cdot x^1 \cdot \frac{(2n)!}{\cancel{x^n}\,(2n+2)!}$$

$$= \left\{\frac{(n+1)\,\cancel{n!}}{\cancel{n!}}\right\}^2 \cdot x \; \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\,\cancel{(2n)!}}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2 x}{(2n+2)(2n+1)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2 x}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\left[1+\frac{1}{n}\right]^2 x}{2\cancel{n}\left[1+\frac{2}{2n}\right] 2\cancel{n}\left[1+\frac{1}{2n}\right]}$$

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$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{[1+1/n]\,x}{4\left[1+\frac{1}{n}\right]\left[1+\frac{1}{2n}\right]}$$

$$= \frac{x\,[1+0]}{4\,[1+0][1+0]} = \frac{x}{4}$$

Thus by ratio test.

Σu_n is convergent if $\frac{x}{4} < 1$ or $x < 4$

Σu_n is divergent if $\frac{x}{4} > 1$ or $x > 4$

and test fails if $\frac{x}{4} = 1$ or $x = 4$

7b) Solve the Bessel's differential Equation leading to $J_n(x)$.

Sol. The Bessel differential Equation of order n is in the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \quad \text{---(1)}$$

where n is a non-negative real const.

$P_0(x) = x^2 \Rightarrow P_0(x) = 0$ at $x = 0$

x is an singular point

Now assume $y = \sum_{r=0}^{\infty} a_r x^{m+r}$ ---(2)

where $a_0 \neq 0$, is solution of Equation (1)

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r)\, x^{m+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1)\, x^{m+r-2}$$

Hence Eqn (1) becomes

$$x^2\left\{\sum_{r=0}^{\infty} a_r (m+r)(m+r-1)\, x^{m+r-2}\right\} + x\left\{\sum_{r=0}^{\infty} a_r (m+r)\, x^{m+r-1}\right\} + (x^2-n^2)\left\{\sum_{r=0}^{\infty} a_r x^{m+r}\right\} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (m+r)(m+r-1)\, x^{m+r} + \sum_{r=0}^{\infty} a_r (m+r)\, x^{m+r} + \sum a_r\, x^{m+r+2} - n^2 \sum_{r=0}^{\infty} a_r\, x^{m+r} = 0$$

$$\sum_{r=0}^{\infty} a_r x^{m+r} \left[(m+r)(m+r-1) + (m+r) - n^2\right] + \sum_{r=0}^{\infty} a_r\, x^{m+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r x^{m+r} \left[(m+r)\{m+r-\cancel{1}+\cancel{1}\} - n^2\right] + \sum_{r=0}^{\infty} a_r\, x^{m+r+2} = 0$$

$$\sum_{r=0}^{\infty} a_r x^{m+r} \left[(m+r)^2 - n^2\right] + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0 \quad \text{--- (3)}$$

Equating the coefficients of x^m [ie $r=0$ in eqn (3)]

$a_0 [m^2 - n^2] = 0$ given that $a_0 \neq 0$

$\Rightarrow m^2 - n^2 = 0 \Rightarrow m^2 = n^2 \Rightarrow m = \pm n$

Equating the coefficient of x^{m+1} in (3) [ie $r = 1$ in eqn (3)]

$a_1[(m+1)^2 - n^2] = 0 \Rightarrow a_1 = 0$

$\because (m+1)^2 - n^2 \neq 0 \quad \because m = \pm n$

Equating the coefficient of x^{m+r} in Eqn(3)

$a_r[(m+r)^2 - n^2] + a_{r-2} = 0$

$a_r[(m+r)^2 - n^2] = -a_{r-2}$

$$a_r = \frac{-a_{r-2}}{(m+r)^2 - n^2}.$$

At $\boxed{m = n} \Rightarrow a_r = -\dfrac{a_{r-2}}{(n+r)^2 - n^2}$

$$a_r = -\frac{a_{r-2}}{\cancel{n^2} + 2nr + r^2 - \cancel{n^2}} = \frac{-a_{r-2}}{2nr + r^2}$$

At $r = 2 \Rightarrow a_2 = -\dfrac{a_0}{4n+4} = -\dfrac{a_0}{4(n+1)}$

At $r = 3 \Rightarrow a_3 = \dfrac{-a_1}{6n+9} \Rightarrow a_3 = 0 \because a_1 = 0$

At $r = 4 \Rightarrow a_4 = \dfrac{-a_2}{8n+16} = -\dfrac{1}{8(n+2)} \cdot \left\{ \dfrac{-a_0}{4(n+1)} \right\}$

$$a_4 = \frac{a_0}{32(n+1)(n+2)}$$

From Eqn (2) taking $m=n$

$$y_1 = x^n[a_0 + a_1 x + a_2 x^2 + \cdots]$$

$$y_1 = x^n\left[a_0 + 0 + \left\{\frac{-a_0}{4(n+1)}\right\}x^2 + 0 + \left\{\frac{a_0}{32(n+1)(n+2)}\right\}x^4 + \cdots\right]$$

$$y_1 = x^n\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \cdots\right]a_0$$

$$y_1 = a_0 x^n\left\{1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \cdots\right\} \quad (4)$$

At $\boxed{m=-n}$

$$y_2 = a_0 x^{-n}\left\{1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} - \cdots\right\} \quad (5)$$

The General Solution of Bessel Eqn is

$$y = C_1 y_1 + C_2 y_2$$

Now choosing $a_0 = \frac{1}{2^n \Gamma n+1}$ in y_1.

$$y_1 = \frac{x^n}{2^n \Gamma n+1}\left\{1 - \left(\frac{x}{2}\right)^2\frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4\frac{1}{(n+1)(n+2)\cdot 2} - \cdots\right\}$$

$$y_1 = \left(\frac{x}{2}\right)^n\left\{\frac{1}{\Gamma n+1} - \left(\frac{x}{2}\right)^2\frac{1}{(n+1)\Gamma n+1} + \left(\frac{x}{2}\right)^4\frac{1}{(n+1)(n+2)\Gamma n+1\cdot 2} - \cdots\right\}$$

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w.k.t $\Gamma n = (n-1)\Gamma n-1$, $\Gamma n+1 = n\Gamma n$

$\Gamma n+2 = (n+1)\Gamma n+1$, $\Gamma n+3 = n+2\,\Gamma n+2 = (n+1)(n+2)\Gamma n+1$

and so on

$$y_1 = \left(\frac{x}{2}\right)^n \left\{ \frac{(-1)^0}{\Gamma n+1\, 0!} + \left(\frac{x}{2}\right)^2 \frac{(-1)^1}{\Gamma n+2\, 1!} + \left(\frac{x}{2}\right)^4 \frac{(-1)^2}{\Gamma n+3 \cdot 2!} + \cdots \right\}$$

$$y_1 = \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma n+r+1 \cdot r!} \left(\frac{x}{2}\right)^{2r} \quad \text{--- (6)}$$

Eqn (6) is called Bessel function of first kind of order r and is denoted by $J_n(x)$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma n+r+1 \cdot r!} \left(\frac{x}{2}\right)^{n+2r}$$

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7c) Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ interms of Legendre's polynomials.

Sol: $x^4 = \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x)$

$x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)$

$x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)$

$x = P_1(x)$

$1 = P_0(x)$

$\Rightarrow f(x) = \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x) +$

$3\left\{ \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) \right\}$.

$- \left\{ \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x) \right\} + 5 P_1(x) + 2 P_0(x)$

$f(x) = \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1(x)$

$+ \frac{28}{15} P_0(x)$

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8a) Test for convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Soln: $u_n = \frac{n^2}{2^n} \Rightarrow u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}[1+1/n]^2}{\cancel{2^n}\cdot 2} \cdot \frac{\cancel{2^n}}{\cancel{n^2}} = \frac{(1+0)^2}{2} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$$

Thus by D'Alemberts ratio test Σu_n is convergent ie $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ is convergent.

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8b) If α and β are two distinct roots of $J_n(x) = 0$. P.T $\int_0^1 x J_n(\alpha x) J_n(\beta x) = 0$ if $\alpha \neq \beta$.

Soln. W.k.t $J_n(\lambda x)$ is a solution of the Eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2) y = 0$$

If $u = J_n(\alpha x)$ $v = J_n(\beta x)$ the associated differential Eqns are

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \quad \text{--- (1)}$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \quad \text{--- (2)}$$

Multiplying (1) by $\frac{v}{x}$ and (2) by $\frac{u}{x}$

we obtain

$$x v u'' + v u' + \alpha^2 u v x - \frac{n^2 u v}{x} = 0$$

$$x u v'' + u v' + \beta^2 u v x - \frac{n^2 u v}{x} = 0$$

On subtracting we obtain

$$x(v u'' - u v'') + (v u' - u v') + (\alpha^2 - \beta^2) u v x = 0$$

$$\frac{d}{dx}\left\{x \cdot (u'v - u v')\right\} = (\beta^2 - \alpha^2)\, x u v.$$

Integrating bothsides w.r.t x betn 0 to 1 we have

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$$\left[x(vu' - uv')\right]_{x=0}^{1} = \beta^2 - \alpha^2 \int_{x=0}^{1} x\,u\,v\,dx$$

ie $$\left[vu' - uv'\right]_{x=1} - 0 = \beta^2 - \alpha^2 \int_0^1 x\,u\,v\,dx \quad (3)$$

since $u = J_n(\alpha x)$ $v = J_n(\beta x)$ we have

$u' = J_n'(\alpha x)\cdot\alpha$ $v' = J_n'(\beta x)\cdot\beta$ and as a consequence of these Eqn (3) becomes

$$\left[J_n(\beta x)\,\alpha\, J_n'(\alpha x) - J_n(\alpha x)\,\beta\, J_n'(\beta x)\right]_{x=1} =$$

$$(\beta^2 - \alpha^2)\int_{x=0}^{1} x\, J_n(\alpha x)\, J_n(\beta x)\,dx$$

Hence $$\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx =$$

$$\frac{1}{\beta^2 - \alpha^2}\left\{\alpha\, J_n(\beta)\, J_n'(\alpha) - \beta\, J_n(\alpha)\, J_n'(\beta)\right\} \quad (4)$$

Since α and β are distinct roots of $J_n(x) = 0$ we have $J_n(\alpha) = 0 = J_n(\beta)$.

with the result the RHS of (4) becomes zero provided $\beta^2 - \alpha^2 \neq 0$ or $\beta \neq \alpha$

Thus we have

$$\int_0^1 x\,J_n(\alpha x)\,J_n(\beta x)\,dx = 0 \quad , \quad \alpha \neq \beta.$$

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8 c) Express $f(x) = x^3 + 2x^2 - x - 3$ interms of Legendre's polynomials.

Sol: $x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)$

$x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)$ $x = P_1(x)$, $1 = P_0(x)$

$\Rightarrow$

$$f(x) = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) + 2\left\{\frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)\right\} - P_1(x) - 3 P_0(x)$$

$$f(x) = \frac{2}{5} P_3(x) + \frac{4}{3} P_2(x) - \frac{2}{5} P_1(x) - \frac{7}{3} P_0(x)$$

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9a) Find the real root of the eqn $x^3 - 2x - 5 = 0$ using Regula - Falsi method, correct to three decimal places.

Soln. $f(x) = x^3 - 2x - 5$

$f(0) = -5$

$f(1) = 1 - 2 - 5 = 1 - 7 = -6$

$f(2) = 8 - 4 - 5 = 8 - 9 = -1 < 0$

$f(3) = 27 - 6 - 5 = 27 - 11 = 16 > 0$

$f(2)\, f(3) < 0$

A real root lies betn $(2, 3)$

I iteration : A root lies betn $(2, 3)$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$a = 2 \qquad b = 3$

$f(a) = -1 \qquad f(b) = 16$

$$x_1 = \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{32 + 3}{16 + 1} = \frac{35}{17} = 2.059$$

$f(x_1) = -0.389$

II iteration : A root lies betn $(2.059, 3)$

$a = 2.059 \qquad b = 3$

$f(b) = -0.389 \qquad f(b) = 16$

$$x_2 = \frac{(2.059)(16) - (3)(-0.389)}{16 - (-0.389)} = 2.081$$

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$f(x_2) = -0.1501$

III iteration A root lies betn $(2.081, 3)$

$a = 2.081$ $b = 3$

$f(a) = -0.1501$ $f(b) = 16$

$$x_3 = \frac{(2.081)(16) - (3)(-0.1501)}{16 - (-0.1501)} = 2.089$$

$f(x_3) = -0.062$

IV iteration : A root lies betn $(2.089, 3)$

$a = 2.089$ $b = 3$

$f(a) = -0.062$ $f(b) = 16$

$$x_4 = \frac{(2.089)(16) - (3)(-0.062)}{16 - (-0.062)} = 2.092$$

$f(x_4) = -0.028$

V iteration : A root lies betn $(2.092, 3)$

$a = 2.092$ $b = 3$

$f(a) = -0.028$ $f(b) = 16$

$$x_5 = \frac{(2.092)(16) - (3)(-0.028)}{16 - (-0.028)} = 2.093$$

$f(x_5) = -0.017$

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VI iteration : A root lies betⁿ (2.093, 3)

$a = 2.093$ $\quad b = 3$

$f(a) = -0.017$ $\quad f(b) = 16$

$$x_6 = \frac{(2.093)(16) - (3)(-0.017)}{16 - (-0.017)} = 2.093$$

$$x_5 = x_6 = 2.093$$

Thus required real root is 2.093

9b) Use Lagrange's interpolating polynomial that approximates the function discribed by the following data.

x	0	1	2	5
$f(x)$	2	3	12	147

Soln Let $x_0 = 0$ $x_1 = 1$ $x_2 = 2$ $x_3 = 5$

$y_0 = 2$ $y_1 = 3$ $y_2 = 12$ $y_3 = 147$

We have Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}[y_0] +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}[y_1] +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}[y_2] +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}[y_3]$$

$$y = f(x) = \frac{(x-1)(x-2)(x-5)[2]}{(-1)(-2)(-5)} + \frac{x(x-2)(x-5)[3]}{(1)(-1)(-4)}$$

$$+ \frac{x(x-1)(x-5)[12]}{(2)(1)(-3)} + \frac{x(x-1)(x-2)[147]}{(5)(4)(3)}$$

$$f(x) = -\frac{1}{5}(x-1)(x-2)(x-5) + \frac{3}{4}x(x-2)(x-5)$$

$$-2x(x-1)(x-5) + \frac{49}{20}x(x-1)(x-2)$$

$$= -\frac{1}{5}\left\{x^3 - 8x^2 + 17x - 10\right\} + \frac{3}{4}\left\{x^3 - 7x^2 + 10x\right\}$$

$$- \left\{2x^3 - 12x^2 + 10x\right\} + \frac{49}{20}\left\{x^3 - 3x^2 + 2x\right\}$$

$$f(x) = x^3\left\{-\frac{1}{5} + \frac{3}{4} - 2 + \frac{49}{20}\right\} + x^2\left\{\frac{8}{5} - \frac{21}{4} + 12 - \frac{147}{20}\right\}$$

$$+ x\left\{-\frac{17}{5} + \frac{30}{4} - 10 + \frac{98}{20}\right\} + \frac{\cancel{10}^{\,2}}{\cancel{5}_{\,1}}$$

$$f(x) = x^3 + x^2 - x + 2.$$

9c) Evaluate $\int_0^1 \frac{x\,dx}{1+x^2}$ by weddle's rule taking seven ordinates, hence find $\log_e 2$

Sol: seven ordinates mean 6 equal parts

$$h = \frac{b-a}{6} = \frac{1-0}{6} = \frac{1}{6}$$

x	0	1/6	2/6	3/6	4/6	5/6	6/6 = 1
$f(x)$	0 y_0	$\frac{6}{37}$ y_1	$\frac{3}{10}$ y_2	$\frac{2}{5}$ y_3	$\frac{6}{13}$ y_4	$\frac{30}{61}$ y_5	$\frac{1}{2}$ y_6

weddle's rule for $n=6$ is given by

$$\int_a^b y\,dx = \frac{3h}{10}\left\{y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right\}$$

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{3}{10}\cdot\frac{1}{6}\left\{0 + 5\cdot\left(\frac{6}{37}\right) + \frac{3}{10} + 6\left(\frac{2}{5}\right) + \frac{6}{13} + 5\left(\frac{30}{61}\right) + \frac{1}{2}\right\}$$

$$\simeq 0.3466$$

Thus $\int_0^1 \frac{x\,dx}{1+x^2} = 0.3466$

To find the value of $\log_e 2$

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

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$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\left[\log_e(1+x^2)\right]_0^1$$

$$= \frac{1}{2}\left[\log_e 2 - \log_e 1\right]$$

$$= \frac{1}{2}\left[\log_e 2 - 0\right]$$

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\cdot\log_e 2 \qquad \because \log_e 1 = 0$$

Equating the theorotical and Numerical value

$$\frac{1}{2}\log_e 2 = 0.3466$$

$$\log_e 2 = 2 \times 0.3466$$

$$\log_e 2 = \underline{0.6932}$$

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10a) Find the real root of the Eqn $xe^x - 2 = 0$ using Newton Raphson method correct to three decimal places

Sol^n. $f(x) = xe^x - 2$

$f(0) = -2 < 0$

$f(1) = 1e^1 - 2 = 0.7183 > 0$

A real root lies bet^n $(0, 1)$ and $x_0 = 1$

$f'(x) = xe^x + e^x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)}$$

$$x_1 = 1 - \frac{0.7183}{5.4365} = 0.8679$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8679 - \frac{f(0.8679)}{f'(0.8679)}$$

$$x_2 = 0.8679 - \frac{0.0673}{4.419} = 0.8527$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.8527 - \frac{f(0.8527)}{f'(0.8527)}$$

$$x_3 = 0.8527 - \frac{4.107 \times 10^{-4}}{4.3463} = 0.8526$$

Thus required root is $\underline{\underline{0.852}}$

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10b) Use Newton's divided difference formula to find $f(4)$ given the data

x	0	2	3	6
$f(x)$	-4	2	14	158

Soln. Divided difference table is as follows.

x	$f(x)$	1st D.D $[x_0\, x_1]$	2nd D.D $[x_0\, x_1, x_2]$	3rd D.D $[x_0\, x_1\, x_2\, x_3]$
0	-4			
		$\frac{2-(-4)}{2-0} = 3$		
2	2		$\frac{12-3}{3-0} = 3$	
		$\frac{14-2}{3-2} = 12$		$\frac{9-3}{6-0} = 1$
3	14		$\frac{48-12}{6-2} = 9$	
		$\frac{158-14}{6-3} = 48$		
6	158			

we have Newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0)[x_0\, x_1] + (x-x_0)(x-x_1)[x_0\, x_1\, x_2] + \cdots$$

$$f(4) = -4 + (4-0)[3] + (4-0)(4-2)[3] + (4-0)(4-2)(4-3)[1]$$

$$= -4 + 12 + 24 + 8 = 40$$

$$f(4) = 40$$

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10 c) Use simpson's $\frac{3}{8}^{th}$ rule to Evaluate $\int_1^4 e^{1/x} dx$

Soln: To apply simpson's $\frac{3}{8}^{th}$ rule n must be multiple of 3 and we shall take $n=3$ itself.

length of Each strip $= \frac{b-a}{n} = \frac{4-1}{3} = \frac{3}{3} = 1$

$\boxed{h=1}$

simpson's $\frac{3}{8}^{th}$ rule for $n=3$ is given by

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0+y_3) + 3(y_1+y_2)\right]$$

x	1	2	3	4
$y=e^{1/x}$	2.7183	1.6487	1.3956	1.2840
	y_0	y_1	y_2	y_3

$$\int_1^4 e^{1/x} dx = \frac{3(1)}{8}\left\{(2.7183 + 1.2840) + 3(1.6487 + 1.3956)\right\}$$

$$\int_1^4 e^{1/x} dx = 4.9257$$