KLS VISHWANATHRAO DESHPANDE INSTITUTE OF TECHNOLOGY

DEPARTMENT OF MATHEMATICS

ADVANCED CALCULUS AND NUMERICAL METHODS (18MAT21)

(Model Question Paper)

SLNO		QUESTIONS	MARKS	BLT
Q. I		**MODULE-01**		
1	a	Find the directional derivative of $\emptyset = x^2yz + 4xz^2$ at (1, –2, –1) along $2i - j - 2k$.	06	L1
	b	Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$.	07	L1
	c	Show that $\vec{F} = (y+z)i + (z+x)j + (x+y)k$ is irrotational. Also find a scalar function $\emptyset$ such that $\vec{F} = \nabla\emptyset$.	07	L2
		OR		
2	a	Find the total work done by the force represented by $\vec{F} = 3xyi - yj + 2zxk$ in moving a particle round the circle $x^2 + y^2 = 4$.	06	L1
	b	Find the area between the parabolas $y^2 = 4x$ *and* $x^2 = 4y$ with the help of Green's theorem in a plane.	07	L1
	c	If $\vec{F} = 2xyi + yz^2j + zxk$ and S is the rectangular parallelopiped bounded by $x = 0, y = 0, z = 0, x = 2, y = 1, z = 3$. Find the flux across S by using Gauss divergence theorem.	07	L2
Q. II		**MODULE-02**		
3	a	Solve $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$.	06	L3
	b	Solve $(D^3 - 1)y = 3cos2x$.	07	L3
	c	Solve $y'' + 2y' + y = 2x + x^2$.	07	L3
		OR		
4	a	Using the method of variation of parameters solve $\frac{d^2y}{dx^2} + y = secx\ tanx$.	06	L3
	b	Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$.	07	L3
	c	Solve $(2x+1)^2y'' - 6(2x+1)y' + 16y = 8(2x+1)^2$.	07	L3
Q. III		**MODULE-03**		
5	a	Form the partial differential equation by eliminating the arbitrary constants from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.	06	L3
	b	Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, for which $\frac{\partial z}{\partial y} = -2\sin y$, when $x = 0$ & $z = 0$ if y is an odd multiple of $\frac{\pi}{2}$.	07	L3
	c	Derive the expression for one dimensional wave equation.	07	L3
6	a	Form the partial differential equation by eliminating the arbitrary function $z = f(y - 2x) + g(2y - x)$.	06	L3

<table>
<tr><td></td><td>b</td><td>Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.</td><td>07</td><td>L3</td></tr>
<tr><td></td><td>c</td><td>Solve one dimensional heat equation, using the method of separation of variables.</td><td>07</td><td>L3</td></tr>
<tr><td>Q.IV</td><td colspan="2">MODULE-04</td><td></td><td></td></tr>
<tr><td>7</td><td>a</td><td>Test for convergence or divergence of the series :
$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots \ldots \ldots ..$$</td><td>06</td><td>L2</td></tr>
<tr><td></td><td>b</td><td>If α and β are two distinct roots of $J_n(x) = 0$ then prove that $\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)dx = 0$ if $\alpha \neq \beta$.</td><td>07</td><td>L2</td></tr>
<tr><td></td><td>c</td><td>Express $x^3 + 2x^2 - 4x + 5$ in terms of Legendre's polynomials.</td><td>07</td><td>L2</td></tr>
<tr><td colspan="5"></td></tr>
<tr><td>8</td><td>a</td><td>Test for convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{(n+1)^n\, x^n}{n^{n+1}}$.</td><td>06</td><td>L2</td></tr>
<tr><td></td><td>b</td><td>With usual notations show that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, sinx$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\, cosx$.</td><td>07</td><td>L2</td></tr>
<tr><td></td><td>c</td><td>Use Rodrigue's formula to show that $P_3(cos\theta) = \frac{1}{8}(3cos\theta + 5cos3\theta)$.</td><td>07</td><td>L2</td></tr>
<tr><td>Q. V</td><td colspan="2">MODULE-05</td><td></td><td></td></tr>
<tr><td>9</td><td>a</td><td>Compute the fourth root of 12 correct to 3 decimal places using Regula Falsi method.</td><td>06</td><td>L2</td></tr>
<tr><td></td><td>b</td><td>From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46
<table>
<tr><td>Age</td><td>45</td><td>50</td><td>55</td><td>60</td><td>65</td></tr>
<tr><td>Premium(in Rs)</td><td>114.84</td><td>96.16</td><td>83.32</td><td>74.48</td><td>68.48</td></tr>
</table></td><td>07</td><td>L2</td></tr>
<tr><td></td><td>c</td><td>Compute the value of $\int_{0.2}^{1.4}(sinx - \log_e x + e^x)dx$ using Simpsons 3/8th rule taking six parts.</td><td>07</td><td>L2</td></tr>
<tr><td colspan="5"></td></tr>
<tr><td>10</td><td>a</td><td>Find a real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places using Newton Raphson method.</td><td>06</td><td>L2</td></tr>
<tr><td></td><td>b</td><td>The following table gives the premium payable at ages in years completed. Interpolate the premium payable at age 35 completed. Use Lagrange's formula.
<table>
<tr><td>Age completed</td><td>25</td><td>30</td><td>40</td><td>60</td></tr>
<tr><td>Premium in Rs.</td><td>50</td><td>55</td><td>70</td><td>95</td></tr>
</table></td><td>07</td><td>L2</td></tr>
<tr><td></td><td>c</td><td>Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by Weddle's rule taking $h = 1/2$.</td><td>07</td><td>L3</td></tr>
</table>

SECOND SEMESTER B.E. Examination
(MODEL QUESTION PAPER)
– ADVANCED CALCULUS AND NUMERICAL METHODS (18MAT21)
DETAILED SOLUTION

1

a) $\phi = x^2yz + 4xz^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$= \frac{\partial}{\partial x}(x^2yz + 4xz^2) i + \frac{\partial(x^2yz+4xz^2)}{\partial y} j + \frac{\partial}{\partial z}(x^2yz+4xz^2).k$$

$$= (2xyz + 4z^2) i + x^2z.j + (x^2y + 8xz)k$$

At the pt. $(1,-2,-1)$

$$\therefore \boxed{(\nabla\phi)_{(1,-2,-1)} = 8i - j - 10k}$$

The unit vector in the direction of $2i-j-2k$ is

$$\hat{n} = \frac{2i-j-2k}{\sqrt{4+1+4}} = \frac{2i-j-2k}{3}$$

$\therefore$ The required directional derivative is

$$\nabla\phi.\hat{n} = (8i-j-10k).\frac{(2i-j-2k)}{3} = \frac{16+1+20}{3}$$

$$\boxed{\nabla\phi.\hat{n} = \frac{37}{3}}$$

b) Let $\phi = x^3+y^3+z^3-3xyz$

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k,$$

$$= \frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)i + \frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)j + \frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)k$$

$$= (3x^2-3yz)\hat{i} + (3y^2-3xz)j + (3z^2-3xy)k$$

$$div\vec{F} = \nabla.\vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right).\vec{F}$$

P.T.O

$\operatorname{div}\vec{F} = 6x+6y+6z = 6(x+y+z)$

$$\operatorname{Curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right] - j\left[\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial yz}(3x^2-3yz)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$= i[-3x+3x] - j[-3y+3y] + k[-3z+3z] = 0.$$

$\therefore \boxed{\operatorname{Curl}\vec{F} = 0}$

c) We have to show that curl $\vec{F} = 0$

Where $\vec{F} = (y+z)i + (z+x)j + (x+y)k$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(y+x) - \frac{\partial}{\partial z}(x+z)\right] - j\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(x+z) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= i(1-1) - j(1-1) + k(1-1) = 0$$

$\therefore \boxed{\operatorname{curl}\vec{F} = 0}$

$\therefore$ $\vec{F}$ is irrotational.

Consider $\vec{F} = \nabla\phi$

$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (y+z)i + (x+z)j + (x+y)k$$

$\frac{\partial \phi}{\partial x} = y+z$. Integrating $\phi = xy+yz+f_1(y,z)$

$\frac{\partial \phi}{\partial y} = x+z$ Integrating $\phi = xy+yz+f_2(x,z)$

$\frac{\partial \phi}{\partial z} = x+y$ Integrating $\phi = xz+yz+f_3(x,y)$

Let $f_1 = yz$, $f_2 = xz$, $f_3 = xy$

$\therefore \boxed{\phi = xy+yz+xz}$

2.

a) Total Work done $W = \int_C \vec{F}\cdot d\vec{r}$

$x^2+y^2=4$ can be represented in parametric form

$x = 2\cos\theta$, $y = 2\sin\theta$ and $z=0$; $0 \le \theta \le 2\pi$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C (3xy\,dx - y\,dy + 2xz\,dz)$$

$$= \int_{\theta=0}^{2\pi} 3(4\cos\theta\sin\theta)(-2\sin\theta)\,d\theta - \int_{\theta=0}^{2\pi} 4\sin\theta\cos\theta\,d\theta$$

$$= -24\int_0^{2\pi} \sin^2\theta\cos\theta\,d\theta - 2\int_0^{2\pi}\sin 2\theta\,d\theta$$

$$= -24\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} + 2\left[\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= -\frac{24}{3}\left[\sin^3(2\pi) - \sin^3(0)\right] + \left[\cos 4\pi - \cos 0\right]$$

$$= -\frac{24}{3}\left[0-0\right] + \left[1-1\right] = 0$$

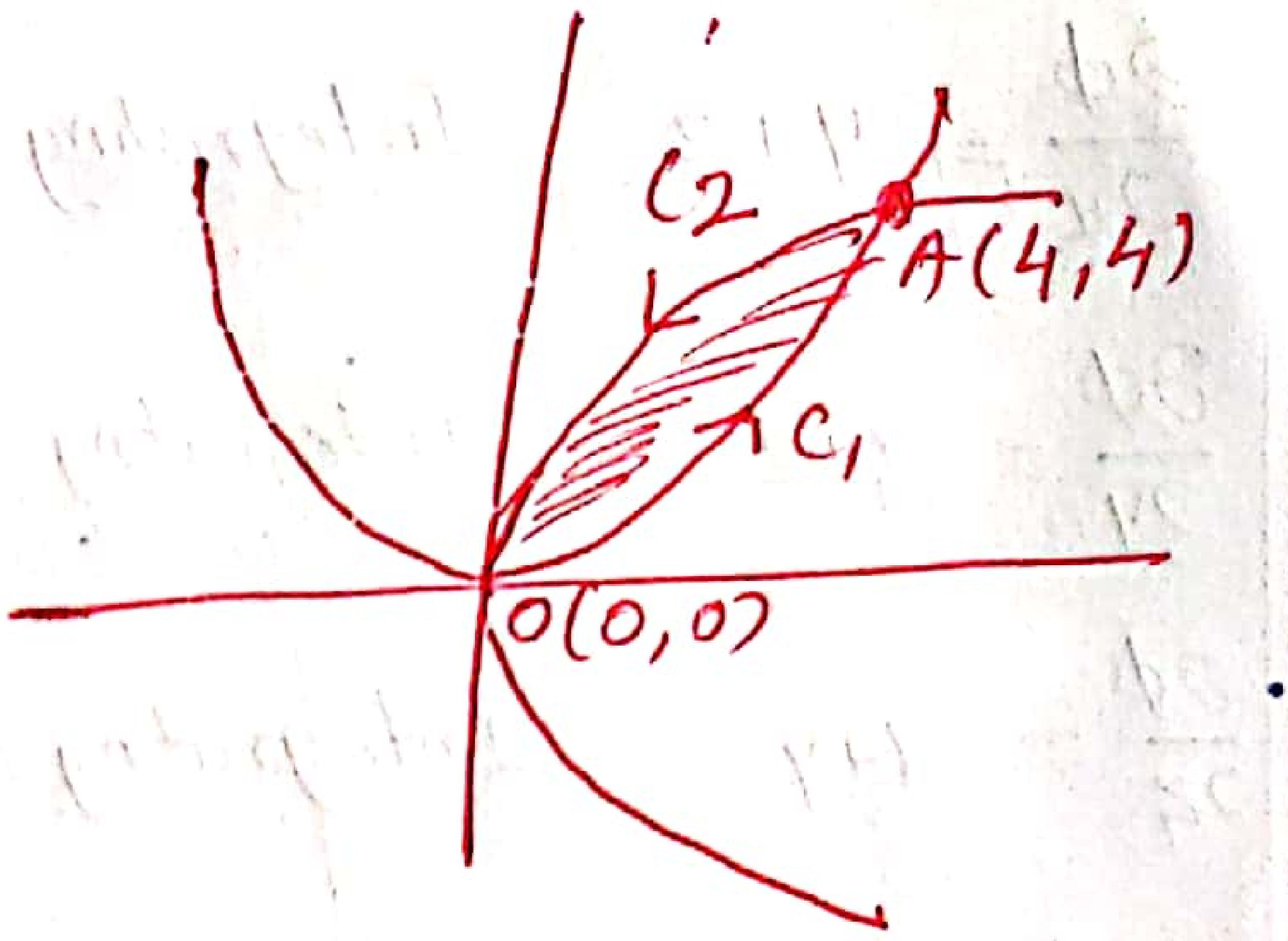
$\therefore$ Thus the total work done $\boxed{W=0}$

b) The Area $A = \iint dx\,dy$

The pt. of intersection of $y^2=4x$ and $x^2=4y$ are O(0,0) & A(4,4)

For $x^2=4y \Rightarrow dy=\frac{x}{2}dx$ & $0\le x\le 4$

For $y^2=4x \Rightarrow dx=\frac{y}{2}dy$ & $4\le y\le 0$.



$$A=\frac{1}{2}\int_{C_1}(xdy-ydx)+\frac{1}{2}\int_{C_2}(xdy-ydx)$$

$$=\frac{1}{2}\int_{x=0}^{4}\frac{x^2}{4}dx+\frac{1}{2}\int_{y=0}^{4}\left(\frac{y^2}{4}\right)dy.$$

$$=\frac{1}{2\times4}\left[\frac{x^3}{3}\right]_0^4+\frac{1}{8}\left[\frac{y^3}{3}\right]_0^4.$$

$$=\frac{1}{24}\left[64+64\right]=\frac{128}{24}=\frac{16}{3}=5.33 \text{ sq. units, ie } \boxed{A=5.33}$$

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e) Flux across 'S' is given by,

$$S=\iint_S \vec{F}\cdot d\vec{s}=\iint_S \vec{F}\cdot\hat{n}\,ds.$$

By the divergence theorem $\iint_S \vec{F}\cdot\hat{n}\,ds=\iiint_V div\,\vec{F}\,dv.$

We have. $\vec{F}=2xyi+yz^2j+zxk.$

$$\therefore div\,\vec{F}=\nabla\cdot\vec{F}=\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)(2xyi+yz^2j+zxk)$$

$$\nabla\cdot\vec{F}=2y+z^2+x$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds=\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3}(2y+z^2+x)dz\,dy\,dx$$

$$=\int_{x=0}^{2}\int_{y=0}^{1}\left[2yz+\frac{z^3}{3}+xz\right]_0^3dy\,dx.$$

$$=\int_{x=0}^{2}\int_{y=0}^{1}(6y+9+3x)\,dy\,dx.$$

$$=\int_{x=0}^{2}(3y^2+9y+3xy)_0^1\,dx.$$

~~5~~ ⑤

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{2}(12+3x)\,dx$$

$$= \left[12x + \frac{3}{2}x^2\right]_{x=0}^{2}.$$

$$= \left[\left(24 + \frac{3}{2}\cdot 4\right) - 0\right] = 30.$$

Thus $\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = 30}$

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3. a) Given $(D^3 - 3D^2 + 11D - 6)y = 0$

A.E is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$m = 1, 2, 3$

∴ The roots are real & distinct.

∴ The general soln is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$

$$\boxed{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}}$$

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b) Given $(D^3 - 1)y = 3\cos 2x$

A.E. is $m^3 - 1 = 0$

$$\Rightarrow (m+1)(m^2+m+1) = 0$$

$m^2 + m + 1 = 0$

$$\Rightarrow m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\therefore\ m = 1,\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\ .$$

The roots are real & Imaginary

$$C.F = y_c = c_1 e^{x} + e^{-x/2}\left[c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x\right]$$

P.I. $y_p = \frac{1}{D^3-1} 3\cos 2x$ Replace D^2 by (-4)

$$y_p = \frac{1}{-4D-1} 3\cos 2x = \frac{-3}{(4D+1)} \frac{(4D-1)}{(4D-1)} \cdot \cos 2x.$$

$$= \frac{-3(4D-1)}{(16D^2-1)_{D\to -4}} \cos 2x = \frac{3}{65}(4D-1)\cos 2x.$$

$$y_p = \frac{-3}{65}[8\sin 2x + \cos 2x]$$

∴ The General soln is $y = y_c + y_p$.

$$\therefore y = c_1 e^{x} + e^{-x/2}[c_2\cos(\sqrt{3}/2)x + c_3\sin(\sqrt{3}/2)x] - \frac{3}{65}[8\sin 2x + \cos 2x] \quad //$$

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a) Given

$$y'' + 2y' + y = 2x + x^2$$

A.E. is $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$\Rightarrow m = -1, -1$

The roots are real & repeated.

$$C.F = (c_1 + c_2 x)e^{-x}$$

$$y_p = \frac{1}{1+2D+D^2} \cdot (2x + x^2)$$

By the method of division.

$$\begin{array}{r|l} & x^2 - 2x + 2 \\ \hline 1+2D+D^2 & x^2 + 2x \\ & x^2 + 4x + 2 \\ \hline & -2x - 2 \\ & -2x - 4 \\ \hline & 2 \\ & 2 \\ \hline & 00 \end{array}$$

$$\therefore y_p = x^2 - 2x + 2$$

∴ General soln is $y = y_c + y_p$

ie $\boxed{y = (c_1 + c_2 x)e^{-x} + x^2 - 2x + 2}$

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4 a) Given $\frac{d^2y}{dx^2} + y = \sec x \cdot \tan x$

A.E is $m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m = \pm i$

The roots are Imaginary.

∴ C.F $= y_c = c_1 \cos x + c_2 \sin x$

Let $y = A(x)\cos x + B(x)\sin x$.

$$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ \overset{\sin x}{\cancel{\cos}} & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$\boxed{W = 1}$

$$A = -\int \frac{\phi_2(x).X}{W} = -\int \tan^2 x \, dx = \int 1 - \sec^2 x \, dx = x - \tan x + k_1$$

$$B = \int \frac{\phi_1(x).X}{W} = \int \tan x \, dx = \log(\sec x) + k_2$$

∴ $y = A\cos x + B\sin x$

ie $y = k_1 \sin x + k_2 \cos x + x\cos x + \sin x\,[\log(\sec x)]$

is the required soln.

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b) put $\log x = t$ or $x = e^t$ ~~ie~~

$(D^2 - 4D + 4)y = 1 + 2e^t + e^{2t}$

A.E is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

The roots are real & distinct

P.T.O

C. F = $y_c = (c_1+c_2t)e^{2t} = (c_1+c_2\log x)x^2$

P.I = $\frac{1}{D^2-4D+4}[1+2e^t+e^{2t}]$

$= \frac{1}{[D^2-4D+4]}_{D\to 0}(1) + 2.\frac{1}{[D^2-4D+4]}_{D\to 1}e^t + \frac{1}{D^2-4D+4}e^{2t}.$

$= \frac{1}{4} + 2.\frac{1}{1-4+4}e^t + \frac{1}{2}t^2e^{2t}.$

$= \frac{1}{4} + 2e^t + t^2e^{2t}. = \frac{1}{4} + 2x + \frac{x^2(\log x)^2}{2}$

∴ The soln is

$y = (c_1+c_2\log x)x^2 + \frac{1}{4} + 2x + \frac{x^2(\log x)^2}{2}$

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c) Given $(2x+1)^2y'' - 6(2x+1)y' + 16y = 8(2x+1)^2$

put $t = \log(2x+1)$ or $2x+1 = e^t$ in the above eqn

∴ $(D^2-4D+4)y = 8e^{2t}$

A.E is $m^2-4m+4=0$

$(m-2)^2=0 \Rightarrow m=2,2.$

The roots are real & repeated.

$y_c = (c_1+c_2t)e^{2t} = [c_1+c_2[\log(2x+1)]](2x+1)^2.$

$y_p = \frac{1}{(D-2)^2}8e^{2t}$ Replace D by 2

$y_p = \frac{1}{(2-2)^2}8e^{2t} = \frac{8}{2}t^2e^{2t} = 4t^2e^{2t}.$

$= 4[\log(2x+1)]^2(2x+1)^2.$

∴ $y = y_c + y_p$

$y = (2x+1)^2[c_1+c_2\{\log(2x+1)\}] + 4\{\log(2x+1)\}^2(2x+1)^2$ // is the required soln.

(9)

5. a) Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \longrightarrow ①$

differentiating ① partially w.r.t. x & y we get

$$\frac{x}{a^2}+\frac{zp}{c^2}=0 \longrightarrow ②$$

$$\frac{y}{b^2}+\frac{zq}{c^2}=0 \longrightarrow ③$$

Diff eqn ② w.r.t x again. we get

$$\frac{1}{a^2}+\frac{1}{c^2}[zr+p^2] \longrightarrow ④$$

From eqn ② $\frac{1}{a^2}=\frac{-zp}{c^2x}$

Substituting this in eqn ④ we get

$$zp=x(zr+p^2)$$

Thus

$z\frac{\partial z}{\partial x}=xz\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2$ is the required P.d.e.

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b) Given $\frac{\partial^2 z}{\partial x\partial y}=\sin x\sin y$

ie $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\sin x\sin y$ which is a Non-Homogeneous P.d.e.

Integrating w.r.t x treating 'y' as constant we get

$$\frac{\partial z}{\partial y}=-\sin y\cos x+f(y) \longrightarrow ①$$

Integrating w.r.t 'y' treating 'x' as constant we get.

$z=\cos x\cos y+F(y)+g(x) \xrightarrow{} ②$ Where $F(y)=\int f(y)\,dy$.

By the data $\frac{\partial z}{\partial y}=-2\sin y$ when $x=0$

From eqn ①

P.T.O

$f(y) = -\sin y$

Hence $F(y) = \int f(y)\,dy = \int -\sin y\,dy = \cos y$

∴ Eqⁿ ② becomes

$z = \cos x \cdot \cos y + \cos y + g(x) \longrightarrow ③$

Using $z=0$ if $y = (2n+1)\pi/2$ in ③ we get

$g(x) = 0$

Thus $z = \cos x \cos y + \cos y$

ie $\boxed{z = \cos y\,[1+\cos x]}$ is the required soln.

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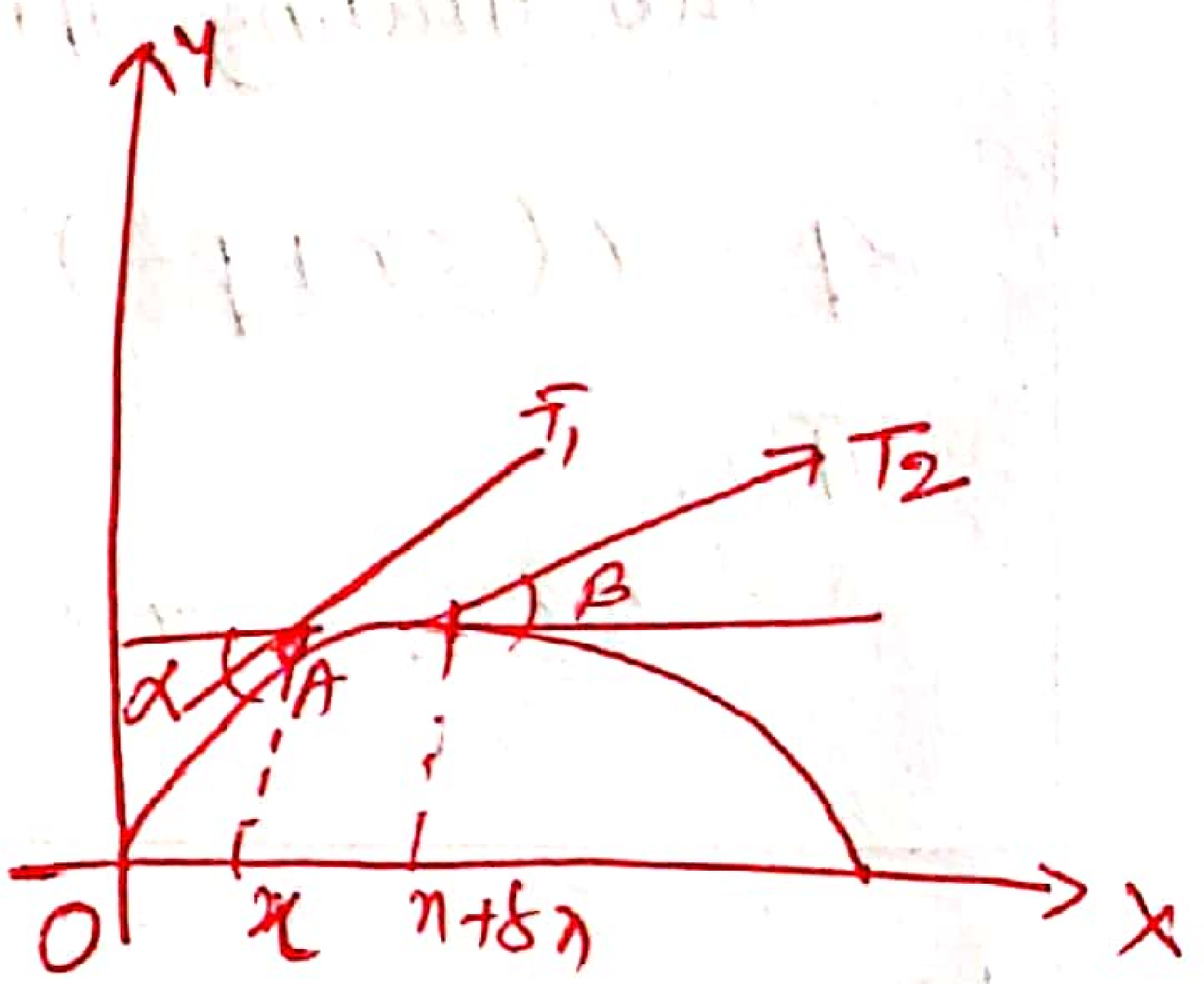
c) Consider a flexible string tightly stretched between two fixed points at a distance 'l' apart. Let 'ρ' be the mass per unit length of the string.



Let T_1 and T_2 be the tension at the points A & B.

$$T_1 \cos\alpha = T_2 \cos\beta = T \longrightarrow ①$$

Applying Newton's second law of motion, that is

$$F = ma$$

$$T_2 \sin\beta - T_1 \sin\alpha = (\rho\, dx)\frac{\partial^2 y}{\partial t^2}$$

Dividing throughout by T.

$$\frac{T_2}{T}\sin\beta - \frac{T_1}{T}\sin\alpha = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 y}{\partial t^2} \longrightarrow ②$$

From ① $\frac{T_1}{T} = 1/\cos\alpha$; $T_2/T = 1/\cos\beta$

using this in eqⁿ ② we get.

ie $\tan\beta - \tan\alpha = \frac{\rho}{T}\,\delta x\,\frac{\partial^2 y}{\partial t^2} \longrightarrow ③$

$\tan\beta = \left(\frac{\partial y}{\partial x}\right)_{x+\delta x}$ and $\tan\alpha = \left(\frac{\partial y}{\partial x}\right)_x$

∴ Eqn ③ becomes

$$\lim_{\delta x \to 0} \frac{(\partial y/\partial x)_{x+\delta x} - (\partial y/\partial x)_x}{\delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

ie $\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial u^2}{\partial t^2}$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$ or $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

or $\boxed{u_{tt} = c^2 u_{xx}}$ Where $c^2 = T/\rho$

---

6. a) Given $z = f(y-2x) + g(\cancel{y}\, 2y-x) \longrightarrow ①$

diff ① partially w.r.t x, y. we get

$p = \frac{\partial z}{\partial x} = -2f'(y-2x) - g'(2y-x) \longrightarrow ②$

$q = \frac{\partial z}{\partial y} = f'(y-2x) + 2g'(2y-x) \longrightarrow ③$

diff. ② partially w.r.t x and y.

$r = \frac{\partial^2 z}{\partial x^2} = 4f''(y-2x) + g''(2y-x) \longrightarrow ④$

$s = \frac{\partial^2 z}{\partial x \partial y} = -2f''(y-2x) - 2g''(2y-x) \longrightarrow ⑤$

$t = \frac{\partial^2 z}{\partial y^2} = f''(y-2x) + 4g''(2y-x) \longrightarrow ⑥$

④ × 2 + ⑤ gives $2r+s = 6f''(y-2x) \longrightarrow ⑦$

⑤ × ② + ⑥ gives $2s+t = -3f''(y-2x)$ —— ⑧

⑦ ÷ ⑧ gives $\frac{2r+s}{2s+t} = -2$ or $2r+5s+2t=0$

Thus

$$2\frac{\partial^2 z}{\partial x^2} + 5\cdot\frac{\partial^2 z}{\partial x \partial y} + 2\cdot\frac{\partial^2 z}{\partial y^2} = 0$$

b) Given $(x^2-y^2-z^2)p + 2xyq = 2xz$

which is of the form $Pp + Qq = R$

The Auxillary eqn is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

ie $$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \longrightarrow ①$$

Consider $\frac{dy}{2xy} = \frac{dz}{2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$ integrating

$\log y = \log z + \log c_1 \Rightarrow \log(y/z) = \log c_1$

ie $y/z = c_1$

Using the multipliers x, y, z each ratio in ①

$$\frac{xdx+ydy+zdz}{x^3-xy^2-xz^2+2xy^2+2xz^2} = \frac{xdx+ydy+zdz}{x^3+xy^2+xz^2} = \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2}$$

consider.

$$\frac{dy}{2xy} = \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$$

or $$\frac{dy}{y} = \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}$$

Integrating we get

$\log(x^2+y^2+z^2) + \log c_2 = \log y.$

or $$\frac{y}{x^2+y^2+z^2} = c_2$$

∴ Thus the general solution is

$$\boxed{\phi(y/z,\ y/x^2+y^2+z^2) = 0}$$

6 c) Consider $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow ①$

Let $u = XT$ where $X = X(x)$ and $T = T(t)$ be the solution of Eqn ①

$\therefore \frac{\partial}{\partial t}(XT) = c^2 \frac{\partial^2}{\partial x^2}(XT)$

or $XT' = c^2 . T . X''$ where $T' = \frac{dT}{dt}$ and $X'' = \frac{d^2x}{dx^2}$

By equating to the constant k, we have.

$\frac{1}{c^2T}\frac{dT}{dt} = k$ and $\frac{1}{X}\frac{d^2X}{dx^2} = k$

or $(D^2 - c^2k)T = 0$ and $(D^2 - k)X = 0$

Case i: Let $k = 0$

$u = XT = C_1(C_2x + C_3)$

or $u(x,t) = Ax + B$ where $A = C_1C_2$ & $B = C_1C_3$

Case ii: Let k be the positive integer $k = p^2$

Hence $u = C_1' e^{c^2p^2t}[C_2'(e^{px}) + C_3'(e^{-px})]$

or $u = e^{c^2p^2t}[A'e^{px} + B'e^{-px}]$ where $A' = C_1'C_2'$, $B' = C_1'C_3'$

Case iii: Let k be the -ve integer say $k = -p^2$

$u = XT = C_1'' e^{-c^2p^2t}[C_2'' \cos px + C_3'' \sin px]$

or $u = e^{-c^2p^2t}[A'' \cos px + B'' \sin px]$

7. a) Test for convergence

Given $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + ----$

P.T.O

$$u_n = \frac{x^{2n}}{(n+2)\sqrt{n+1}} \qquad u_{n+1} = \frac{x^{2(n+1)}}{(n+3)\sqrt{n+2}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{2(n+1)}}{(n+3)\sqrt{n+2}} \cdot \frac{(n+2)\sqrt{n+1}}{x^{2n}} = \frac{n+2}{n+3}\sqrt{\frac{n+1}{n+2}} \cdot x^2.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+2}{n+3} \cdot \sqrt{\frac{n+1}{n+2}}$$

$$= \lim_{n\to\infty} \frac{n(1+2/n)}{n(1+3/n)} \sqrt{\frac{n(1+1/n)}{n(1+2/n)}} \cdot x^2 = x^2$$

Thus by D'Alemberts ratio test.

Σu_n is Convergend if $x^2<1$

Σu_n is divergent if $x^2>1$

and test fails if $x^2=1$

---

b) W.K.T. $J_n(\lambda x)$ is a solution of eq^n

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0$$

If $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ the associated D.E are

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \longrightarrow ①$$

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \longrightarrow ②$$

multiplying ① by $\frac{u}{x}$ and ② by $\frac{u}{x}$ we get.

$$xvu'' + vu' + \alpha^2uvx - n^2 \cdot \frac{uv}{x} = 0$$

$$xuv'' + uv' + \beta^2uvx - n^2\frac{uv}{x} = 0$$

substracting both we get

$$x(vu'' - uv'') + (vu' - uv') + (\alpha^2 - \beta^2)uvx = 0$$

$$\frac{d}{dx}\{x(vu' - uv')\} = (\beta^2 - \alpha^2)xuv.$$

$$[x(vu'-uv')]_{x=0}^{1} = (\beta^2-\alpha^2)\int_{x=0}^{1} xuv\,dx.$$

$$[vu'-uv'] = (\beta^2-\alpha^2)\int_{0}^{1} xuv\,dx \longrightarrow ③$$

$u = J_n(\alpha x)$, $v = J_n(\beta x)$

$u' = \alpha J_n'(\alpha x)$, $v' = \beta J_n'(\beta x)$ Eqⁿ ③ becomes

$$J_n(\beta x)\alpha J_n'(\alpha x) - J_n(\alpha x).\beta J_n'(\beta x)] = (\beta^2-\alpha^2)\int_{x=0}^{1} x J_n(\alpha x) J_n(\beta x)\,dx.$$

$$\Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \frac{1}{\beta^2-\alpha^2}[\alpha J_n(\beta) J_n'(\alpha) - \beta J_n(\alpha) J_n'(\beta)] \longrightarrow ④$$

since α and β are distinct roots of $J_n(x)=0$ we have $J_n(\alpha)=0=J_n(\beta)$. R.H.S. of eqⁿ ④ becomes zero provided $\beta^2-\alpha^2 \neq 0$ iè $\alpha \neq \beta$.

$$\therefore \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$$

c) Let $f(x) = x^3 + 2x^2 - 4x + 5$

$P_0(x) = 1$ $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

$x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$

$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x).$

$x = P_1(x)$, $P_0(x) = 1$

$$\Rightarrow f(x) = \left\{\frac{2}{5}P_3(x) + \frac{43}{3}P_1(x)\right\} + 2\left\{\frac{1}{3}P_0(x) + \frac{2}{3}P_0(x)\right\} - 4P_1(x) + 5P_0(x).$$

$$f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{17}{5}P_1(x) + \frac{17}{3}P_0(x)$$

8 a) Test for Convergence

$u_n = \frac{(n+1)^n x^n}{n^{n+1}}$, $u_n^{1/n} = \frac{\{(n+1)^n\}^{1/n}\{x^n\}^{1/n}}{\{n^{n+1}\}^{1/n}.}$

$(u_n)^{1/n} = \frac{(n+1).x}{n^{(1+1/n)}} = \frac{(n+1)x}{n.n^{1/n}}$

$\lim_{n\to\infty}\{u_n\}^{1/n} = \lim_{n\to\infty}\frac{(n+1)x}{n.n^{1/n}} = \lim_{n\to\infty}\frac{\cancel{n}(1+1/n).x}{\cancel{n}.n^{1/n}}$

$\lim_{n\to\infty}(u_n)^{1/n} = x$ $[\because \lim_{n\to\infty}(n^{1/n}) = 1]$

∴ Thus by cauchy's root test Σu_n is Convergent if $x<1$ and divergent if $x>1$ and the test fails if $x=1$

---

b) WKT. $J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma n+r+1 . r!} \longrightarrow ①$

Put $n = 1/2$

$J_{1/2}(x) = \sum_{r=0}^{\infty}(-1)^r.\left(\frac{x}{2}\right)^{\frac{1}{2}+2r}\frac{1}{\Gamma r+3/2 . r!}$

On Expanding we get

$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(3/2)} - \left(\frac{x}{2}\right)^2\frac{1}{\Gamma(5/2).1!} + \left(\frac{x}{2}\right)^4\frac{1}{\Gamma(7/2).2!} - \cdots\right] \longrightarrow ②$

WKT $\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(3/2) = (\Gamma 1/2)\frac{1}{2} = \frac{\sqrt{\pi}}{2}$

$\Gamma(5/2) = \frac{3}{2}.\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$

$\Gamma(7/2) = \frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma(b/2) = \frac{15}{8}\sqrt{\pi}$

Substituting in eqn ②

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{4}{3\sqrt{\pi}} + \frac{x^4}{16}\cdot\frac{8}{15\sqrt{\pi}\cdot 2} - - - -\right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2\pi}}\left[2 - \frac{x^2}{3} + \frac{x^4}{60} - - - -\right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2\pi}}\cdot\frac{2}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{120} - - - -\right]$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{6} + \frac{x^5}{120} - - - -\right]$$

$$\boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x}$$

Put $n = \frac{-1}{2}$ in eqn ①

$$J_{-1/2}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{\frac{-1}{2}+2r}\cdot\frac{1}{\Gamma\left(r+\frac{1}{2}\right)\cdot r!}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\left[\frac{1}{\Gamma(1/2)} - \left(\frac{x}{2}\right)^2\cdot\frac{1}{\Gamma(3/2)\cdot 1!} + \left(\frac{x}{2}\right)^4\frac{1}{\Gamma(5/2)\cdot 2!} - - -\right]$$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4}\cdot\frac{2}{\sqrt{\pi}} + \frac{x^4}{16}\cdot\frac{4}{3\sqrt{\pi}\cdot 2} - - - -\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - - - -\right]$$

$$\boxed{J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x}$$

---

c) WKT. $P_3(x) = \frac{1}{2}[5x^3 - 3x]$

$$P_3[\cos\theta] = \frac{1}{2}[5\cos^3\theta - 3\cos\theta]$$

WKT $\cos^3\theta = \frac{1}{4}[\cos 3\theta + 3\cos\theta]$

$$\therefore P_3[\cos\theta] = \frac{1}{2}\left[\frac{5}{4}[\cos 3\theta + 3\cos\theta] - 3\cos\theta\right.$$

P.T.O

Completion, SMS to

parents

$$P_3(\cos\theta) = \frac{1}{8}[5\cos 3\theta + 3\cos\theta]$$

$$\therefore P_3[\cos\theta] = \frac{1}{8}[3\cos\theta + 5\cos 3\theta]_{//}$$

9. a) Let $x = \sqrt[4]{12}$

$\therefore x^4 - 12 = 0$

Let $f(x) = x^4 - 12$

$f(0) = -12, \; f(1) = -11 < 0, \; f(2) = 4 > 0$

A real root of $f(x)=0$ lies in $(1, 2)$ and will be in the neighbourhood of 2.

$f(1.7) = -3.6479, \; f(1.8) = -1.5024 < 0, \; f(1.9) = 1.0321 > 0$

$\therefore$ The root lies in the interval. $(1.8, 1.9)$

Let $a = 1.8, \; f(a) = -1.5024, \; b = 1.9, \; f(b) = 1.0321 > 0$

$\therefore$ By regula falsi method

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(1.8)(1.0321) - (1.9)(-1.5024)}{1.0321 + 1.5024}$$

$x_1 = 1.8593$

$f(x_1) = f(1.8593) = (1.8593)^4 - 12 = -0.0492$

$\therefore$ root lies in the Interval $(\underset{a}{1.8593}, \underset{b}{1.9})$

$$x_2 = \frac{(1.8593)(1.0321) - (1.9)(-0.0492)}{1.0321 + 0.0492} = 1.8612$$

$f(x_2) = f(1.8612) = (1.8612)^4 - 12 = -0.00025 < 0$

$\therefore$ The root lies in the interval $(1.8612, 1.9)$

$$x_3 = \frac{1.8612(1.0321) - (1.9)(-0.00025)}{1.0321 + 0.00025.} = 1.86121$$

$\therefore$ Thus the 2nd & 3rd Iterations are identical upto ~~4~~ 3. decimal places. Therefore the required 4th root of 12 is

$x = 1.861$

b) Preparing the Forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 45$	$y_0 = 114.84$	Δy_0 -18.68	$\Delta^2 y_0$ 5.84	$\Delta^3 y_0$ -1.84	$0.68 - \Delta^4 y_0$
50	96.16	-12.84	4	-1.16	
55	83.32	-8.84	2.84		
60	74.48	-6			
65	68.48				

Let $x_p = 46 = x_0 + ph$

$$\Rightarrow p = \frac{x_p - x_0}{h} = \frac{46-45}{5} = 0.2$$

By Newton's Forward Interpolation formula.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y(46) = 114.84 + (0.2)(-18.68) + \frac{(0.2)(0.2-1)\cancel{(0.2-2)}}{2!}(5.84) +$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)}{3!}(-1.84) + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{4!} \times (0.68)$$

Solving

$$\boxed{y(46) = 110.53.}$$

---

c) $h = \frac{b-a}{n} = \frac{1.4-0.2}{6} = 0.2$

Let $y = f(x) = \sin x - \log_e x + e^x$

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y	3.0295	2.7975	2.8976	3.1660	3.5598	4.0698	4.7042
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ rule.

$$I = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + \cancel{8}\, 2y_3 + 3y_4 + 3y_5 + y_6\right]$$

$$= \frac{3 \times 0.2}{8}\left[3.0295 + (3 \times 2.7975) + (3 \times 2.8976)\right] + (2 \times 3.1660) + (3 \times 3.5598) +$$

$$+ (3 \times 4.0698) + 4.7042\Big]$$

$\therefore I = 4.053$

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a) $f(x) = x^3 - 2x - 5, \quad f'(x) = 3x^2 - 2$

$f(0) = -5 < 0, \; f(1) = -6 < 0, \; f(2) = -1 < 0, \; f(3) = 16 > 0$

A real root lies in the (2, 3). Let $x_0 = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{(2)^3 - (2\times2) - 5}{(3\times2^2) - 2} = 2.1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \frac{(2.1)^3 - (2\times2.1) - 5}{(3\times2.1^2) - 2} = 2.0946.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0946 - \frac{(2.0946)^3 - (2\times2.0946) - 5}{[3\times(2.0946)^2] - 2}$$

$x_3 = 2.0946$

Thus the required approximate root correct to '3' decimal places is 2.0946.

b) By Lagranges interpolation fromula.

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 +$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

~~f(35) = (35-25)(10)~~

$$f(35) = \frac{5(-5)(-25)\,50}{(-5)(-15)(-35)} + \frac{10(-5)(-25)(55)}{5(-10)(-30)} + \frac{10(5)(-25)(70)}{15(10)(-20)}$$

$$+ \frac{(10)(5)(-5)}{(35)(30)(20)} \times 95$$

$$f(35) = \frac{31250}{-2625} + \frac{68750}{1500} + 1\cancel{25}\;\frac{87500}{3000} - \frac{23750}{21000}$$

$$f(35) = 61.96.$$

c) Length of the interval $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$, $n=6$

The points of division are $x = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$

x	0	1/6	1/3	1/2	2/3	5/6	1
$y = x/1+x^2$	0 y_0	6/37 y_1	3/10 y_2	2/5 y_3	6/13 y_4	30/61 y_5	1/2 y_6

By Weddle's rule.

$$\int_a^b y\,dx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

$$\int_0^6 \frac{x}{1+x^2}\,dx = \frac{3}{60}\left[0+(5\times 6/37)+3/10+\left(6\times\frac{2}{5}\right)+\frac{6}{13}+\left(5\times\frac{30}{61}\right)+\frac{1}{2}\right]$$

$$= 0.3466$$

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Hande 26/7/2021.

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