Model Question Paper-2 with effect from 2019-20 (CBCS Scheme)

USN | | | | | | | | | | | **18MAT41**

Fourth Semester B.E. Degree Examination
Complex Analysis, Probability and Statistical Methods

(Common to all Programmes)

Time: 3 Hrs Max.Marks: 100

Note: Answer any FIVE full questions, choosing at least ONE question from each module. Use of statistical tables allowed.

Module-1

1 (a) Show that $w = f(z) = \log z \quad (z \neq 0)$is analytic, using Cauchy-Riemann equations. **(06 Marks)**

(b) Derive Cauchy-Riemann equations in polar form. **(07Marks)**

(c) Find the analytic function$f(z) = u + iv$, given $u + v = x + y + e^x(\cos y + \sin y)$. **(07Marks)**

OR

2 (a) Show that real and imaginary parts of an analytic function are harmonic. **(06 Marks)**

(b) If $f(z)$ is a regular function of z, show that$\{(\partial/\partial x)|f(z)|\}^2 + \{(\partial/\partial y)|f(z)|\}^2 = |f'(z)|^2$. **(07Marks)**

(c) If $w = f(z) = \varphi(x, y) + i\psi(x, y)$ represents the complex potential of an electrostatic field, where $\psi = (x^2 - y^2) + [x/(x^2 + y^2)]$, find $f(z)$ and hence determine φ. **(07Marks)**

Module-2

3 (a) State and prove Cauchy's integral formula. **(06 Marks)**

(b) Discuss the transformation $w = f(z) = e^z$ with respect to the lines parallel to the co-ordinate axes in z-plane. **(07Marks)**

(c) Find the bilinear transformation which maps the points $z = \infty, i, 0$ into the points $w = -1, -i, 1$ respectively. Also, find the fixed points of the transformation. **(07Marks)**

OR

4 (a) Verify Cauchy's theorem for $\int_c \frac{1}{z} dz$ where c is the triangle with the vertices (1, 2), (3,2) and (1,4). **(06 Marks)**

(b) Evaluate: $\int_c \frac{dz}{z^2-4}$ over the curve C: $|z + 2| = 1$. **(07Marks)**

(c) Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = 2, i, -2$ respectively. What are the invariant points under this transformation? **(07Marks)**

Module-3

5 (a) (06 Marks)

The probability distribution of a random variable X is given by the following table:

$X(=x_i)$	0	1	2	3	4	5
$P(X)$	k	$5k$	$10k$	$10k$	$5k$	k

Find (i) the value of k (ii) $P(x \leq 1)$ (iii)$P(0 \leq x < 3)$.

(b) In a certain city, the duration of the shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for (i) 10 minutes or more (ii) less than 10 minutes (iii) between 10 to 12 minutes? (07Marks)

(c) The marks of 1000 students in an examination follow a normal distribution with mean 70 and standard deviation 5. Find the number of the students whose marks will be (i) less than 65 (ii) more than 75 (iii) between 65 and 75. (07Marks)

OR

6 (a) The probability density function of a random variable $X(= x)$ is $f(x) = \begin{cases} kx^2, & -3 < x < 3 \\ 0, & otherwise \end{cases}$. (06 Marks)

Find (i) the value of k (ii) $P(1 \leq x \leq 2)$ and (iii) $P(x > 1)$.

(b) Suppose 300 misprints are randomly distributed throughout a book of 500 pages, find the probability that a given page contains (i) exactly three misprints (ii) less than three misprints and (iii) four or more misprints. (07Marks)

(c) The I.Q. of students in a certain college is assumed to be normally distributed with mean 100 and variance 25. If two students are selected at random, find the probability that (i) both of them have I.Q. between 102 and 110 (ii) at least one of them have I.Q .between 102 and 110 (iii) at most one of them have I.Q. between 102 and 110. (07Marks)

Module-4

7 (a) Ten competitors in a beauty contest are ranked by two judges A and B in the following order: (06 Marks)

ID No. of competitors	1	2	3	4	5	6	7	8	9	10
Judge A	1	6	5	3	10	2	4	9	7	8
Judge B	6	4	9	8	1	2	3	10	5	7

Calculate the rank correlation coefficient

(b) Fit a best fitting parabola $y = ax^2 + bx + c$ for the following data: **(07Marks)**

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

(c) With usual notation, compute $\bar{x}, \bar{y}$ and r from the following lines of regression: **(07Marks)**
$2x + 3y + 1 = 0$ and $x + 6y - 4 = 0$.

OR

8 (a) If θ is the acute angle between the lines of regression, then show that $\tan\theta = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\left(\frac{1-r^2}{r}\right)$. Explain the significance when $r = 0$ & $r = \pm 1$. **(06 Marks)**

(b) The following table gives the heights of fathers(x) and sons (y): **(07Marks)**

x	65	66	67	67	68	69	70	72
y	67	68	65	68	72	72	69	71

Calculate the coefficient of correlation and lines of regression.

(c) A simply supported beam carries a concentrated load P at its midpoint. Corresponding to various values of P, the maximum deflection Y is measured and is given below: **(07Marks)**

P	100	120	140	160	180	200
Y	0.45	0.55	0.55	0.70	0.80	0.85

Find a best fitting straight line in the form $Y = a + bP$ to the above data and hence estimate Y when $P = 150$.

Module-5

9 (a) The joint probability distribution of desecrate random variables X and Y is given below: **(06 Marks)**

X \ Y	1	3	6
1	1/9	1/6	1/18
3	1/6	1/4	1/12
6	1/18	1/12	1/36

Determine (i) marginal distribution of X and Y (ii) Are X and Y statistically independent?

(b) A sample of 100 students is taken from a large population. The mean height of the students in this sample is 160cm. Can it be reasonably regarded that in the population the mean height is 165cm and the standard deviation is 10cm at 5% level of significance?. **(07Marks)**

(c) A random sample of 10 boys had the following I.Q. :70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Does this data support the assumption of a population mean I.Q. of 100 at 5% level of significance? ($t_{0.05}$ for 9 d.f. is 2.262). **(07Marks)**

OR

10 (a) Explain the terms: (i) Null hypothesis (ii) Confidence intervals (iii) Type-I and Type-II errors **(06 Marks)**

(b) A stenographer claims that she can type at the rate of 120 words per minute. Can we reject her claim on the basis of 100 trails in which she demonstrates a mean of 116 words with a standard deviation of 15 words? Use 5% level of significance. **(07Marks)**

(c) Four coins are tossed 100 times and the following results were obtained: **(07Marks)**

No. of Heads	0	1	2	3	4
Frequency	5	29	36	25	5

Fit a binomial distribution for the data and test the goodness of fit ($\chi^2_{0.05}$ for 4 d.f. is 9.49).

MODEL QUESTION PAPER-2 SCHEME

Subject: Complex Analysis, Probability & Statistical Methods (18MAT41)

Module-1

1a. Show that $\omega = f(z) = \log z$ $(z \neq 0)$ is analytic, using Cauchy-Reimann equations.

Sol^n: Given, $\omega = \log z$. Taking, $z = re^{i\theta}$ we have

$$u + iv = \log(re^{i\theta}) = \log r + \log(e^{i\theta})$$

$$= \log r + i\theta \log_e e$$

$$u + iv = \log r + i\theta \qquad \text{since, } \log_e e = 1$$

$$u = \log r \qquad v = \theta$$

$$\frac{\partial u}{\partial r} = \frac{1}{r} \qquad v_r = 0$$

$$\frac{\partial u}{\partial \theta} = 0 \qquad v_\theta = 1$$

Cauchy-Reimann equations in the polar form $ru_r = v_\theta$ and $rv_r = -u_\theta$ are satisfied.

Thus, $\omega = \log z$ is analytic.

1 b. Derive Cauchy-Reimann equations in polar form.

Sol^n: Statement: If, $f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$ is analytic at a point z, then there exists four continuous first order partial derivatives
$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfy the equations:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof: Let, $f(z)$ be analytic at a point $z = re^{i\theta}$ and hence, by definition,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

exists and is unique. In the polar form $f(z) = u(r,\theta) + i\,v(r,\theta)$ and let δz be the increment in z corresponding to the increments $\delta r, \delta\theta$ in r, θ.

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + i v(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + i v(r,\theta)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} + \lim_{\delta z \to 0} i\,\frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \longrightarrow (1)$$

Consider, $z = re^{i\theta}$, Since, z is a function of two variables r, θ we have

$$\delta z = \frac{\partial z}{\partial r}\,\delta r + \frac{\partial z}{\partial \theta}\,\delta\theta$$

$$= \frac{\partial}{\partial r}(re^{i\theta})\,\delta r + \frac{\partial}{\partial \theta}(re^{i\theta})\,\delta\theta$$

$$\delta z = e^{i\theta}\,\delta r + i r e^{i\theta}\,\delta\theta$$

Since, δz tends to zero, we have the following two possibilities.

Case (i): Let, $\delta\theta = 0$, so that $\delta z = e^{i\theta}\delta r$ and $\delta z \to 0$ imply $\delta r \to 0$.

Now (1) becomes,

$$f'(z) = \lim_{\delta\theta\to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{ire^{i\theta}\,\delta\theta} + i\lim_{\delta\theta\to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{ire^{i\theta}\,\delta\theta}$$

$$= \frac{1}{ire^{i\theta}}\left[\lim_{\delta\theta\to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{\delta\theta} + i\lim_{\delta\theta\to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{\delta\theta}\right]$$

$$f'(z) = \frac{1}{ire^{i\theta}}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right] = \frac{1}{re^{i\theta}}\left[\frac{1}{i}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right]$$

But, $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$ and hence we have

$$f'(z) = \frac{1}{re^{i\theta}}\left[-i\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta}\right] = e^{-i\theta}\left[-\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right]$$

Therefore, $$f'(z) = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right] \longrightarrow (3)$$

Equating the RHS of (2) & (3) we have,

$$e^{-i\theta}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right]=e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta}-\frac{i}{r}\frac{\partial u}{\partial \theta}\right]$$

Cancelling $e^{-i\theta}$ on both sides and equating the real and imaginary parts, we get

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

or $\quad r u_r = v_\theta \quad$ and $\quad r v_r = -u_\theta$

Thus, we have established Cauchy-Reimann equations in the polar form.

(c) Find the analytic function $f(z)=u+iv$, given $u+v = x+y+e^x(\cos y+\sin y)$.

Solⁿ: Given, $u+v = x+y+e^x(\cos y+\sin y)$

Differentiating partially w.r.t. x and y

$$u_x+v_x = 1+e^x(\cos y+\sin y) \longrightarrow (1)$$

$$u_y+v_y = 1+e^x(\cos y-\sin y) \longrightarrow (2)$$

put $x=z$ and $y=0$ in eqns (1) & (2),

$$u_x+v_x = 1+e^z, \quad u_y+v_y = 1+e^z \longrightarrow (3)$$

Using C-R equations for the LHS $u_y=-v_x$ and $v_y=u_x$ we have, eqn (3)

$$-v_x+u_x = 1+e^z \longrightarrow (4)$$

Solving, $\quad u_x+v_x=1+e^z, \quad u_x-v_x=1+e^z$

Adding, $\quad 2u_x = 2+2e^z \Rightarrow u_x = 1+e^z$

Subtracting, $\quad 2v_x = 0 \quad \Rightarrow v_x = 0$

We know, $f'(z) = u_x + i v_x$

$\Rightarrow f'(z) = 1 + e^z$

Integrating w.r.t. 'z',

$f(z) = \int 1\, dz + \int e^z dz + c$

$\underline{\underline{f(z)}} = z + e^z + c$, is the required analytic function.

OR

2a. Show that real and imaginary parts of an analytic function are harmonic.

Solⁿ: Cartesian form: Let $f(z) = u + iv$ be analytic

We have to show that u and v satisfy Laplace's

Since, f(z) is analytic we have Cauchy-Reimann

equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ $\longrightarrow$ (1)

$\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ $\longrightarrow$ (2)

Differentiating (1) w.r.t. x and (2) w.r.t. y partially we get,

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y}$ and $\dfrac{\partial^2 v}{\partial y \partial x} = -\dfrac{\partial^2 u}{\partial y^2}$

But, $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$ is true & we have

$\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}$ or $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u$ is harmonic

Again differentiating (1) w.r.t. 'y' & (2) w.r.t. 'x' partially we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$$

But, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ is true & we have,

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow v \text{ is harmonic}$$

Thus, both real and imaginary parts of an analytic function in the cartesian form satisfy Laplace's equation in the cartesian form.

(b) If $f(z)$ is a regular function of z, show that

$$\left\{ \frac{\partial}{\partial x} |f(z)| \right\}^2 + \left\{ \frac{\partial}{\partial y} |f(z)| \right\}^2 = |f'(z)|^2.$$

Sol^n: Let, $f(z) = u + iv$ be the regular function.

$$|f(z)| = \sqrt{u^2+v^2} = \phi$$

We have to prove that, $\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = |f'(z)|^2$

or $\phi_x^2 + \phi_y^2 = |f'(z)|^2$, where $\phi = \sqrt{u^2+v^2}$

Consider, $\phi^2 = u^2 + v^2$ (squaring ϕ)

Differentiating w.r.t. 'x' partially we get,

$2\phi\phi_x = 2uu_x + 2vv_x$ and dividing by 2

$$\phi\phi_x = uu_x + vv_x \longrightarrow (1)$$

Differentiating $\phi^2 = u^2+v^2$, partially w.r.t. 'y'

$$2\phi\phi_y = 2uu_y + 2vv_y$$

i.e. $$\phi\phi_y = uu_y + vv_y \longrightarrow (2)$$

Squaring and adding (1) & (2) we have,

$$\phi^2(\phi_x^2+\phi_y^2) = (uu_x+vv_x)^2 + (vv_y+uu_y)^2$$

$$= u^2u_x^2 + 2uvu_xv_x + v^2v_x^2$$

$$+ u^2u_y^2 + 2uvu_yv_y + v^2v_y^2$$

Since, $f(z) = u+iv$ is analytic, we have

C-R equations : $u_y = -v_x$ and $v_y = u_x$

By using these in the second terms of RHS,

$$\phi^2(\phi_x^2+\phi_y^2) = (u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x)$$

$$+ (u^2v_x^2 + v^2u_x^2 - 2uvu_xv_x)$$

$$= u^2(u_x^2+v_x^2) + v^2(u_x^2+v_x^2)$$

$$= (u^2+v^2)(u_x^2+v_x^2)$$

But, $\phi^2 = u^2+v^2$

$$\phi^2(\phi_x^2+\phi_y^2) = \phi^2(u_x^2+v_x^2)$$

$$\therefore \phi_x^2+\phi_y^2 = u_x^2+v_x^2 \longrightarrow (3)$$

But, $f'(z) = u_x + iv_x$

$\therefore |f'(z)| = \sqrt{u_x^2+v_x^2}$ or $|f'(z)|^2 = u_x^2+v_x^2$

Using in the RHS of (3) we get, $\phi_x^2+\phi_y^2 = |f'(z)|^2$

(c) If $\omega = f(z) = \phi(x,y) + i\psi(x,y)$ represents the complex potential of an electrostatic field, where $\psi = (x^2-y^2) + \left[\frac{x}{x^2+y^2}\right]$, find $f(z)$ & hence determine ϕ.

Soln: Given, $\psi = (x^2-y^2) + \left[\frac{x}{x^2+y^2}\right]$

$$\psi_x = 2x + \frac{(x^2+y^2)1 - x.2x}{(x^2+y^2)^2} = 2x + \frac{(y^2-x^2)}{(x^2+y^2)^2}$$

$$\psi_y = -2y + \frac{(x^2+y^2)0 - x.2y}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Consider, $f'(z) = \phi_x + i\psi_x$. But, $\phi_x = \psi_y$

putting $x = z,\ y = 0$

$$f'(z) = \psi_y + i\psi_x$$

$$f'(z) = 0 + i\left(2z + \frac{(-z^2)}{(z^2)^2}\right) = i\left(2z - \frac{1}{z^2}\right)$$

$$\therefore f(z) = i\int\left(2z - \frac{1}{z^2}\right)dz + c = i\left(z^2 + \frac{1}{z}\right) + c$$

Thus, $f(z) = i\left(z^2 + \frac{1}{z}\right) + c$

To find ϕ, we have to separate the RHS into real and imaginary parts.

$$\phi + i\psi = i\left\{ (x+iy)^2 + \frac{1}{x+iy} \right\} + c$$

$$= i\left\{ (x^2 + i^2y^2 + 2xiy) + \frac{x-iy}{(x+iy)(x-iy)} \right\} + c$$

$$\phi + i\psi = i\left\{ (x^2-y^2) + 2xiy \right\} + i\left\{ \frac{x-iy}{x^2+y^2} \right\} + c$$

$$= i(x^2-y^2) - 2xy + \frac{ix}{x^2+y^2} + \frac{y}{x^2+y^2} + c$$

$$\therefore \phi + i\psi = \left(-2xy + \frac{y}{x^2+y^2} \right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2} \right) + c$$

Equating the real and imaginary parts we observe that the imaginary part is same as the given problem and the required real part is

$$\boxed{\phi = -2xy + \frac{y}{x^2+y^2}}$$

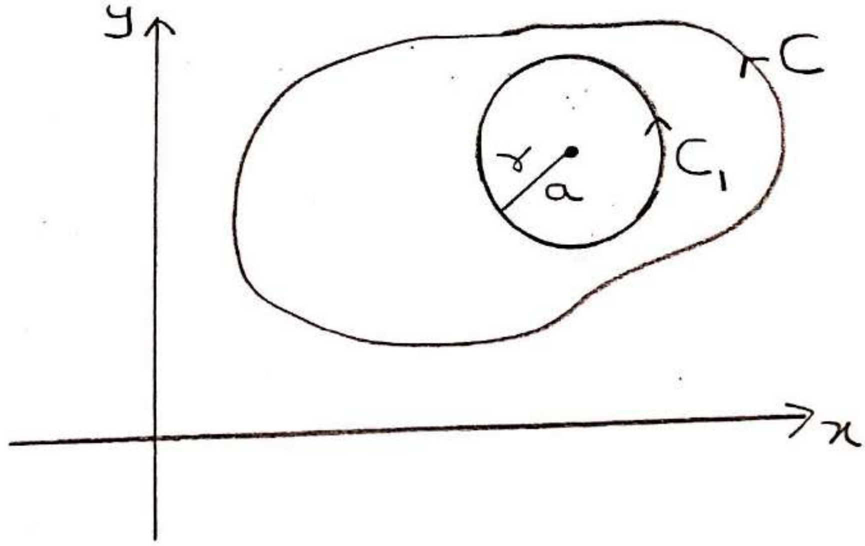
MODULE - 2

3 (a) State and prove Cauchy's integral formula.

Solⁿ: Statement: If $f(z)$ is analytic inside and on a simple closed curve 'C' and if 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

Proof: Since 'a' is a point within 'C', we shall enclose it by a circle C_1 with $z=a$ as centre and r as radius such that C_1 lies entirely within C. The function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the annular region between C and C_1.



Now, by a property

$$\int_C \frac{f(z)}{z-a}\, dz = \int_{C_1} \frac{f(z)}{z-a}\, dz$$

The equation of C_1 (circle with centre 'a' and radius 'r') can be written as $|z-a| = r$.

That is,

$$z - a = re^{i\theta}$$

or $$z = a + re^{i\theta}, \quad 0 \le \theta \le 2\pi,$$

$$dz = ire^{i\theta}\, d\theta$$

Using these results in the RHS of (1), we have

$$\int_C \frac{f(z)}{z-a}\, dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\, d\theta$$

$$\int_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a+re^{i\theta})\, d\theta$$

$$\int_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a)\, d\theta$$

$$= i f(a) \left[\theta\right]_0^{2\pi} = 2\pi i f(a)$$

Thus, $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\, dz$$

Thus, the proof of Cauchy's Integral formula.

3b. Discuss the transformation $\omega = f(z) = e^z$ with respect to the lines parallel to the co-ordinate axes in z-plane.

Solⁿ: The co-ordinate axes in the z-plane are represented by $x=0$, $y=0$.

First we discuss the transformation $\omega = e^z$

Consider, $\omega = e^z$

$$u+iv = e^{x+iy} = e^x \cdot e^{iy}$$

$$u+iv = e^x (\cos y + i \sin y)$$

$$\therefore u = e^x \cos y, \quad v = e^x \sin y \longrightarrow (1)$$

We will find the image in the ω-plane corresponding to the straight lines parallel to the co-ordinate axes in the z-plane. That is x = constant and y = constant.

Let us eliminate x and y separately from (1).

Squaring and adding we get

$$u^2+v^2 = e^{2x} \longrightarrow (2)$$

Also by dividing we get, $\frac{v}{u} = \frac{e^x \sin y}{e^x \cos y}$

$$\frac{v}{u} = \tan y \longrightarrow (3)$$

Case-1 : Let $x = c_1$ where c_1 is a constant.

Equation (2) becomes $u^2+v^2 = e^{2c_1}$ = constant $= r^2$

This represents a circle with centre origin and

$z=0,\ \omega=1;\ 1=\dfrac{0+b}{0+d}$

$$b-d=0 \longrightarrow (3)$$

Adding eqn (1) and (2),

$$(1+i)a+b+id=0 \longrightarrow (4)$$

Solving eqns (3) & (4),

$$0a+1b-1d=0 \longrightarrow (3)$$

$$(1+i)a+1b+id=0 \longrightarrow (4)$$

Applying the rule of cross multiplication,

$$\frac{a}{\begin{vmatrix}1 & -1\\ 1 & i\end{vmatrix}} = \frac{-b}{\begin{vmatrix}0 & -1\\ 1+i & i\end{vmatrix}} = \frac{d}{\begin{vmatrix}0 & 1\\ 1+i & 1\end{vmatrix}}$$

$$\frac{a}{i+1} = \frac{-b}{1+i} = \frac{d}{-(1+i)} \quad \text{or} \quad \frac{a}{1}=\frac{b}{-1}=\frac{d}{-1}$$

$\therefore a=1,\ b=-1,\ d=-1$

Also, from (1) $c=-a \quad \therefore c=-1$

Substituting these values of a, b, c, d the assumed bilinear transformation becomes

$$\omega = \frac{1.z-1}{-1.z-1}$$

Thus, $\omega = \dfrac{1-z}{1+z}$ is the required bilinear transformation.

Further, the invariant points are obtained by taking $\omega = z$.

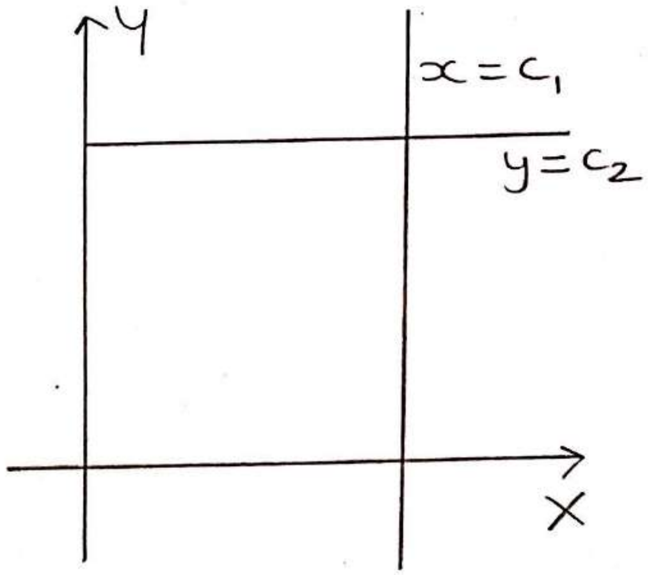
radius r in the ω-plane.

Case-2 : Let, $y = c_2$ where c_2 is a constant.

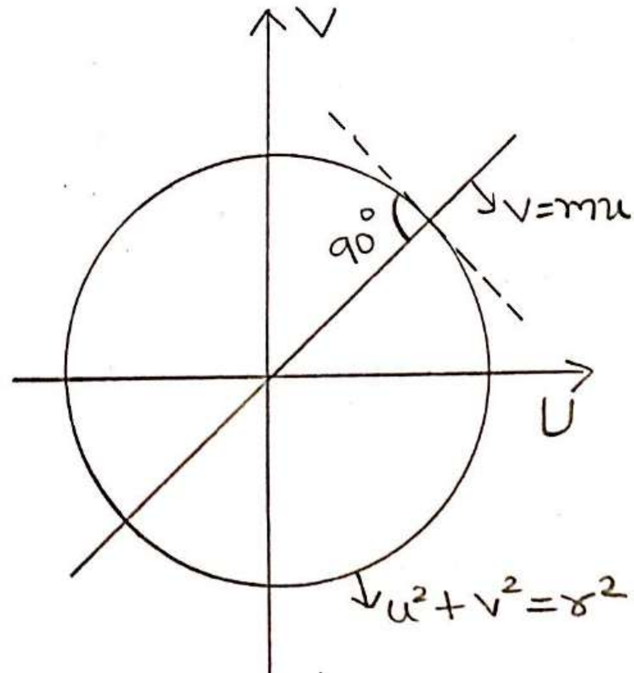
Equation (3) becomes $\frac{u}{v} = \tan c_2 = m$

$$\therefore v = mu$$

This represents a straight line passing through the origin in the ω-plane.



z-plane



ω-plane

Now, using $x = 0$, $y = 0$.

When $y = 0$, eqn (3) becomes $\frac{v}{u} = \tan 0$ or $v = 0$

$\therefore$ the x-axis in the z-plane is mapped onto the u-axis in the ω-plane.

When, $x = 0$, eqn (2) becomes $u^2 + v^2 = 1$

$\therefore$ The y-axis in the z-plane is mapped onto a unit circle with centre origin in the ω-plane.

Conclusion: The straight line parallel to the x-axis $(y=c_2)$ in the z-plane maps onto a straight line passing through the origin in the w-plane. The straight line parallel to the y-axis $(x=c_1)$ in the z-plane maps onto a circle with centre origin and radius r where $r=e^{c_1}$ in the w-plane.

3c. Find the bilinear transformation which maps the points $z=\infty, i, 0$ into the points $w=-1,-i,1$ respectively. Also, find the fixed points of the transformation.

Solⁿ: Let, $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

$z=\infty$, $w=-1$; the bilinear transformation is to be written in the form

$$w = \frac{z\left[a+\frac{b}{z}\right]}{z\left[c+(d/z)\right]} = \frac{a+(b/z)}{c+(d/z)}$$

$$\therefore\ -1 = \frac{a+0}{c+0} \quad \left(\because \frac{1}{z}=0 \text{ when } z=\infty\right)$$

i.e. $$a+c=0 \longrightarrow (1)$$

$$z=i,\ w=-i; \quad -i = \frac{ai+b}{ci+d}$$

$$ai+b-c+di=0 \longrightarrow (2)$$

Scanned with CamScanner

$$z = \frac{1-z}{1+z} \quad \text{or} \quad z(1+z) = 1-z$$

$$z + z^2 = 1 - z$$

$$z^2 + z + z - 1 = 0$$

$$\Rightarrow \quad z^2 + 2z - 1 = 0$$

Factorising,

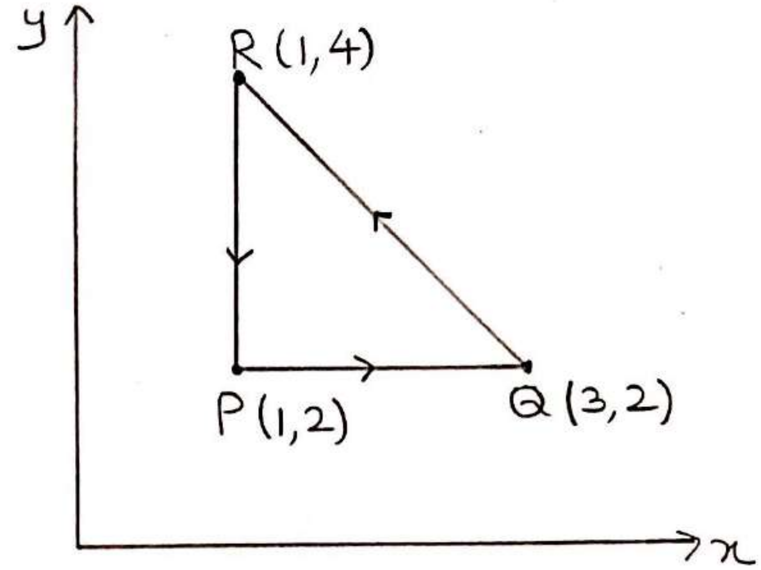
$$z = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2}$$

$$z = -1 \pm \sqrt{2}$$

Thus, the invariant points are $-1 \pm \sqrt{2}$ i.e. $-1+\sqrt{2}$ and $-1-\sqrt{2}$.

4a. Verify Cauchy's theorem for $\int_C \frac{1}{z}\,dz$ where C is the triangle with the vertices $(1,2)$, $(3,2)$ and $(1,4)$.

Sol^n: Here, the given contour C is the triangle in the complex plane as shown in figure below.



We note that $f(z) = 1/z$ is analytic everywhere in the complex plane except at the origin $(z=0)$. Since the origin is outside the given C, $f(z)$ is analytic on and within C. Therefore, by Cauchy's theorem,

$$\int_C f(z)\, dz = \int_C \frac{1}{z}\, dz = 0 \longrightarrow (i)$$

This fact has to be verified. We note that,

$$\int_C \frac{1}{z}\, dz = \int_{PQ} \frac{1}{z}\, dz + \int_{QR} \frac{1}{z}\, dz + \int_{RP} \frac{1}{z}\, dz \longrightarrow (ii)$$

On PQ, we have $y=2$ and $1 \leq x \leq 3$. Therefore,

$$\int_{PQ} \frac{1}{z}\, dz = \int_1^3 \left(\frac{1}{x+2i}\right) dx$$

$$= \log(3+2i) - \log(1+2i) \longrightarrow (iii)$$

Next, we note that the Cartesian equation of QR is

$$\frac{y-2}{x-3} = \frac{4-2}{1-3} \Rightarrow y-2 = -(x-3)$$

or

$$y = -x+5$$

Therefore, the parametric equations of QR may be taken as $x=t$, $y=-t+5$, with $x=t$ varying from 3 to 1. Hence

$$\int_{QR} \frac{1}{z}\, dz = \int_{3}^{1} \frac{d\{t+i(-t+5)\}}{t+i(-t+5)}$$

$$= \log\{t+i(-t+5)\} \Big|_{t=3}^{1}$$

$$= \log(1+4i) - \log(3+2i) \longrightarrow \text{(iv)}$$

Lastly, on RP, we have $x=1$ and y decreases from 4 to 2.

Therefore,

$$\int_{RP} \frac{1}{z}\, dz = \int_{4}^{2} \frac{i\,dy}{(1+iy)} = \log(1+iy) \Big|_{y=4}^{2}$$

$$= \log(1+2i) - \log(1+4i) \longrightarrow \text{(v)}$$

Substituting (iii)-(v) into (ii), we get

$$\int_{C} \frac{1}{z}\, dz = 0$$

This, verifies the result (i)

Thus, for the given $f(z)$ and the given C, the Cauchy's theorem is verified.

Scanned with CamScanner

4b) Evaluate $\int_C \frac{dz}{z^2-4}$ over the curve $C: |z+2| = 1$.

Solⁿ: We note that,

$$\frac{1}{z^2-4} = \frac{1}{(z-2)(z+2)}$$

Resolving $\frac{1}{(z-2)(z+2)}$ into partial fractions.

$$\frac{1}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$$

$$1 = A(z+2) + B(z-2)$$

put $z = 2 \Rightarrow 1 = A(2+2) + B(2-2)$

$$1 = 4A \Rightarrow A = 1/4$$

$z = -2 \Rightarrow 1 = A(-2+2) + B(-2-2)$

$$1 = A(0) + B(-4)$$

$$1 = 0 - 4B$$

$$\Rightarrow -4B = 1$$

$$\Rightarrow B = -1/4$$

$$\therefore \frac{1}{(z-2)(z+2)} = \frac{1}{4} \times \frac{1}{z-2} - \frac{1}{4} \times \frac{1}{z+2}$$

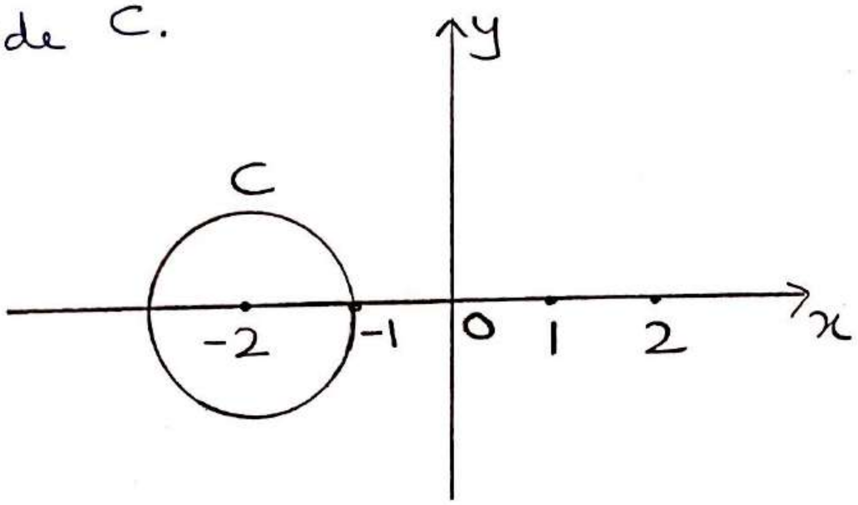
$$= \frac{1}{4}\left(\frac{1}{z-2} - \frac{1}{z+2}\right)$$

Therefore,

$$\int_C \frac{dz}{z^2-4} = \frac{1}{4}\int_C \frac{dz}{z-2} - \frac{1}{4}\int_C \frac{dz}{z+2} \longrightarrow \text{(i)}$$

Scanned with CamScanner

If C is the circle $|z+2|=1$, which is shown in Figure, the point -2 lies inside C and the point 2 lies outside C.



Therefore,

$$\int_C \frac{dz}{z+2} = \int_C \frac{dz}{z-(-2)} = 2\pi i$$

and

$$\int_C \frac{dz}{z-2} = 0$$

Eqn (i) yeilds in this case,

$$\int_C \frac{dz}{z^2-4} = \frac{1}{4}(0-2\pi i) = -\frac{\pi}{2} i$$

Thus,

$$\int_C \frac{dz}{z^2-4} = -\frac{\pi}{2} i$$

4c. Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $\omega = 2, i, -2$ respectively. What are the invariant points under this transformation?

Sol^n: The required transformation is given by,

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega-\omega_3)(\omega_2-\omega_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \longrightarrow (1)$$

Let, $z_1 = 1, z_2 = i, z_3 = -1, \omega_1 = 2, \omega_2 = i, \omega_3 = -2$

Substituting the given values in eqn (1),

$$\frac{(\omega-2)(i+2)}{(\omega+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{\omega-2}{\omega+2} = \frac{(i-2)(i+1)}{(i+2)(i-1)} \times \frac{z-1}{z+1}$$

$$= \frac{i^2+i-2i-2}{i^2-i+2i-2} \times \frac{z-1}{z+1}$$

$$= \frac{-3-i}{-3+i} \times \frac{z-1}{z+1} = p$$

This gives, $\omega - 2 = p(\omega+2)$

$$\omega - 2 = p\omega + 2p$$

$$\omega - \omega p = 2p + 2$$

$$\omega(1-p) = 2(p+1)$$

$$\omega = \frac{2(1+p)}{1-p}$$

$$= \frac{2\left\{1 + \frac{-3-i}{-3+i} \times \frac{z-1}{z+1}\right\}}{1 - \left(\frac{-3-i}{-3+i} \times \frac{z-1}{z+1}\right)}$$

$$= \frac{2\left\{\frac{(-3+i)(z+1) - (3+i)(z-1)}{(-3+i)(z+1)}\right\}}{\frac{(-3+i)(z+1) + (3+i)(z-1)}{(-3+i)(z+1)}}$$

$$= \frac{2\{-3z - 3 + zi + i - 3z + 3 - iz + i\}}{-3z - 3 + iz + i + 3z - 3 + iz - i}$$

$$= \frac{2\{-6z + 2i\}}{-6 + 2iz} = \frac{2(-6z+2i)}{2(iz-3)}$$

$$= \frac{-6z+2i}{iz-3}$$

This is the required bilinear transformation.

For a fixed point, we have $\omega = z$. Setting $\omega = z$ in (i), we get

$$z = \frac{-6z+2i}{iz-3}$$

$z(iz-3) = -6z+2i$

or $\quad iz^2+3z-2i=0$

$z^2+\frac{3z}{i}-2=0$ (Dividing by i)

$z^2+\frac{i}{i}\times\frac{3z}{i}-2=0$

$z^2+\frac{3zi}{i^2}-2=0 \Rightarrow z^2-3zi-2=0$

(since $i^2=-1$)

Solving, $z^2-3zi-2=0$ which is a quadratic equation

$$z=\frac{1}{2}\left\{3i\pm\sqrt{(3i)^2-4\times1\times(-2)}\right\}$$

$$=\frac{1}{2}\left\{3i\pm\sqrt{9i^2+8}\right\}$$

$$=\frac{1}{2}\left\{3i\pm\sqrt{-9+8}\right\}$$

$$=\frac{1}{2}\left\{3i\pm\sqrt{-1}\right\}=\frac{1}{2}\left\{3i\pm i\right\}$$

$$=\frac{1}{2}(3\pm1)i=\frac{1}{2}(3+1)i,\ \frac{1}{2}(3-1)i$$

$$=2i,\ i$$

Thus, $2i$ and i are the fixed points of the given transformation.

5 a) The probability distribution of a random variable X is given by the following table

$X (= x_i)$	0	1	2	3	4	5
$P(X)$	k	5k	10k	10k	5k	k

Find (i) the value of k (ii) $P(x \leq 1)$ (iii) $P(0 \leq x < 3)$

Solⁿ: The properties of probability distribution is

$$P(x) \geq 0 \text{ and } \Sigma P(x) = 1$$

Hence, we must have $k \geq 0$ and

$$k + 5k + 10k + 10k + 5k + k = 1$$

$$32k = 1$$

$$\Rightarrow k = \frac{1}{32}$$

(ii) $P(x \leq 1) = P(0) + P(1) = k + 5k = \frac{1}{32} + \frac{5}{32}$

$$= \frac{6}{32} = 0.1875$$

(iii) $P(0 \leq x < 3) = P(0) + P(1) + P(2)$

$$= k + 5k + 10k$$

$$= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} = \frac{16}{32}$$

b) In a certain city, the duration of the shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for (i) 10 minutes or more (ii) less than 10 minutes (iii) between 10 to 12 minutes?

Solⁿ: The probability density function of the exponential distribution is given by,

$$f(x) = \alpha e^{-\alpha x}, \quad x>0 \text{ and the mean} = \frac{1}{\alpha}$$

$$\frac{1}{\alpha} = 5 \quad \therefore \alpha = \frac{1}{5} \quad \text{and hence } f(x) = \frac{1}{5} e^{-x/5}$$

(i) $$P(x \geq 10) = \int_{10}^{\infty} \frac{1}{5} \cdot e^{-x/5}\, dx = -\left[e^{-x/5}\right]_{x=10}^{\infty}$$

$$\therefore P(x \geq 10) = -(0 - e^{-2}) = e^{-2} = 0.1353$$

(ii) $$P(x<10) = \int_{0}^{10} \frac{1}{5} e^{-x/5}\, dx = -\left[e^{-x/5}\right]_{0}^{10}$$

$$P(x<10) = -(e^{-2}-1) = 1-e^{-2} = 0.8647$$

(iii) $$P(10<x<12) = \int_{10}^{12} \frac{1}{5} \cdot e^{-x/5}\, dx$$

$$= -\left[e^{-x/5}\right]_{10}^{12}$$

$$\therefore P(10<x<12) = -\left(e^{-12/5} - e^{-2}\right) = \underline{\underline{0.0446}}$$

5c. The marks of 1000 students in an examination follow a normal distribution with mean 70 and standard deviation 5. Find the number of the students whose marks will be (i) less than 65 (ii) more than 75 (iii) between 65 and 75.

Solⁿ: Let x represent the marks of students.

Given, $\mu = 70$, $\sigma = 5$. Hence, $z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$

(i) If $x = 65$, $z = -1$ and we have to find $P(z < -1)$

$$P(z<-1) = P(z>1) = P(z \geq 0) - P(0 \leq z \leq 1)$$
$$= 0.5 - \phi(1) = 0.5 - 0.3413$$
$$= 0.1587$$

(i) number of students scoring less than 65 marks
$$= 1000 \times 0.1587 = 158.7 \simeq \underline{\underline{159}}$$

(ii) If $x = 75$, $z = 1$ and we have to find $P(z>1)$

$$P(z>1) = P(z \geq 0) - P(0 \leq z \leq 1) = 0.5 - \phi(1)$$
$$= 0.5 - 0.3413 = 0.1587$$

$\therefore$ number of students scoring marks more than 75 marks $= 1000 \times 0.1587 = 158.7 \simeq \underline{\underline{159}}$

(iii) $P(-1<z<1) = 2P(0<z<1)$
$$= 2\phi(1) = 2(0.3413) = 0.6826$$

$\therefore$ the number of students scoring marks between 65 and 75 $= 1000 \times 0.6826 = 682.6 \simeq \underline{\underline{683}}$

6a. The probability density function of a random variable $X(=x)$ is $f(x) = \begin{cases} kx^2, & -3 < x < 3 \\ 0, & \text{otherwise} \end{cases}$

Find the (i) value of k (ii) $P(1 \leq x \leq 2)$ and (iii) $P(x > 1)$.

Solⁿ: $f(x) \geq 0$ if $k \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{-3} f(x)\,dx + \int_{-3}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{-3}^{3} f(x)\,dx + 0 = 1$$

$$\int_{-3}^{3} kx^2\,dx = 1 \Rightarrow k \int_{-3}^{3} x^2\,dx = 1$$

$$\therefore\ k\left[\frac{x^3}{3}\right]_{x=-3}^{3} = 1 \Rightarrow \frac{k}{3}\left[3^3 - (-3)^3\right] = 1$$

$$\Rightarrow \frac{k}{3}\left[27 - (-27)\right] = 1 \Rightarrow \frac{k}{3}\left[27 + 27\right] = 1$$

$$\Rightarrow \frac{k}{3} \times 54 = 1 \Rightarrow 18k = 1 \Rightarrow \boxed{k = \frac{1}{18}}$$

(ii) $P(1 \leq x \leq 2) = \int_{1}^{2} kx^2\,dx = \frac{1}{18}\int_{1}^{2} x^2\,dx$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{x=1}^{2} = \frac{1}{18} \times \frac{1}{3}\left[2^3 - 1^3\right] = \frac{1}{54}[8-1]$$

$$= \frac{7}{54}$$

(iii) $P(x>1) = \int_1^3 \frac{1}{18} x^2\,dx = \frac{1}{18}\int_1^3 x^2\,dx$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{x=1}^{3} = \frac{1}{54}\left[3^3 - 1^3\right] = \frac{1}{54}\left[27-1\right]$$

$$= \frac{26}{54} = \underline{\underline{\frac{13}{27}}}.$$

(b) Suppose 300 misprints are randomly distributed throughout a book of 500 pages, find the probability that a given page contains (i) exactly three misprints (ii) less than three misprints and (iii) four or more misprints.

Sol^n: We use Poisson distribution. (since n is large)

Mean $(m) = \frac{300}{500} = 0.6$

Poisson distribution, $P(x) = \frac{e^{-m} m^x}{x!}$

(i) $P(x=3) = \frac{e^{-0.6}(0.6)^3}{3!} = \frac{0.5488 \times 0.216}{6}$

$$= 0.0197568 \simeq 0.0197$$

(ii) $P(x<3) = P(x=0) + P(x=1) + P(x=2)$

$$= \frac{e^{-0.6}(0.6)^0}{0!} + \frac{e^{-0.6}(0.6)^1}{1!} + \frac{e^{-0.6}(0.6)^2}{2!}$$

$$= e^{-0.6}\left[1 + 0.6 + \frac{(0.6)^2}{2}\right]$$

$\because 0! = 1$
$1! = 1$
$2! = 2$

$= 0.5488\,[1+0.6+0.18]$

$= \underline{0.976864}$

(iii) $P(x \geq 4) = 1 - P(x<4)$

$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3)]$

$$= 1 - \left[e^{-m}\frac{(0.6)^0}{0!} + e^{-m}\frac{(0.6)^1}{1!} + e^{-m}\frac{(0.6)^2}{2!} + e^{-m}\frac{(0.6)^3}{3!} \right]$$

$$= 1 - e^{-m}\left[1 + 0.6 + \frac{(0.6)^2}{2} + \frac{(0.6)^3}{6} \right]$$

$= 1 - 0.5488\,[1+0.6+0.18+0.036]$

$= 1 - 0.9966 = \underline{3.4 \times 10^{-3}}$ or $\underline{0.0034}$

6c. The I.Q. of students in a certain college is assumed to be normally distributed with mean 100 and variance 25. If two students are selected at random, find the probability that (i) both of them at I.Q. between 102 and 110 (ii) atleast one of them have I.Q between 102 and 110 (iii) atmost one of them have I.Q. between 102 and 110.

Solⁿ: Let x represent the students I.Q.

Given, $\mu = 100$, $\sigma^2 = 25 \quad \therefore \sigma = \sqrt{25} = 5$

The standard normal variate, $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-100}{5}$

(i) We have to find $P(102 < x < 110)$

When $x = 102$, $z = \frac{102-100}{5} = \frac{2}{5} = 0.4$

When $x = 110$, $z = \frac{110-100}{5} = 2$

$$P(0.4 \le z \le 2) = P(0 \le z \le 2) - P(0 \le z \le 0.4)$$
$$= \phi(2) - \phi(0.4)$$
$$= 0.4772 - 0.1554 \text{ [Refer Normal Probability table]}$$
$$= 0.3218$$

$p = 0.3218$, $q = 1 - p = 1 - 0.3218 = 0.6782$

$\therefore$ Probability that both the students have I.Q. between 102 and 110 is $(0.3218)^2 = 0.10355524$

$$\simeq 0.1036$$

(ii) The probability that atleast one of them have I.Q. between 102 and 110 is (by using Binomial distribution) $= P(x \ge 1) = 1 - P(x < 1)$

$$= 1 - P(x = 0)$$
$$= 1 - [{}^2C_0 p^0 q^{2-0}]$$
$$= 1 - [1 \times 1 \times (0.6782)^2]$$
$$= 1 - 0.4599 = 0.5401$$

(iii) The probability that atmost one of them have IQ between 102 and 110 is

$$P(x \le 1) = 1 - P(x > 1)$$
$$= 1 - P(x = 2) = 1 - [{}^2C_2 p^2 q^{2-2}]$$
$$= 1 - [1 \times (0.3218)^2 \times 1]$$

$P(x \leq 1) = 1 - 0.1036 = \underline{\underline{0.8964}}$

MODULE - 4

7 a) Ten competitors in a beauty contest are ranked by two judges A and B in the following order:

ID. No. of Competitors	1	2	3	4	5	6	7	8	9	10
Judge A	1	6	5	3	10	2	4	9	7	8
Judge B	6	4	9	8	1	2	3	10	5	7

Calculate the rank correlation coefficient.

Sol^n: We have, $\rho = 1 - \dfrac{6 \sum d^2}{n(n^2-1)}$

Given data, $n = 10$

$$\therefore \sum d^2 = (1-6)^2 + (6-4)^2 + (5-9)^2 + (3-8)^2 + (10-1)^2 + (2-2)^2 + (4-3)^2 + (9-10)^2 + (7-5)^2 + (8-7)^2$$

$$= 25 + 4 + 16 + 25 + 81 + 0 + 1 + 1 + 4 + 1$$

$$= 158$$

$$\rho = 1 - \frac{6(158)}{10(10^2-1)} = 0.042$$

Thus, $\boxed{\rho = 0.042}$

7b. Fit a best fitting parabola $y = ax^2 + bx + c$ for the following data:

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Sol^n: The normal equations of a parabola are

$$\Sigma y = a\Sigma x^2 + b\Sigma x + nc$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2 \quad , \quad n = 9$$

x	y	xy	x^2y	x^2	x^3	x^4
1	2	2	2	1	1	1
2	6	12	24	4	8	16
3	7	21	63	9	27	81
4	8	32	128	16	64	256
5	10	50	250	25	125	625
6	11	66	396	36	216	1296
7	11	77	539	49	343	2401
8	10	80	640	64	512	4096
9	9	81	729	81	729	6561
$\Sigma x = 45$	$\Sigma y = 74$	$\Sigma xy = 421$	$\Sigma x^2 y = 2771$	$\Sigma x^2 = 285$	$\Sigma x^3 = 2025$	$\Sigma x^4 = 15333$

The normal equation becomes,

$$9c + 45b + 285a = 74$$

$$45c + 285b + 2025a = 421$$

$$285c + 2025b + 15333a = 2771$$

on solving, $a = -0.26732$, $b = 3.58316$, $c = -0.92857$

Thus, the required parabola of fit is,

$$y = -0.267x^2 + 3.52316x - 0.92857$$

7c. With usual notation, compute $\bar{x}$, $\bar{y}$ and r from the following lines of regression:

$2x+3y+1=0$ and $x+6y-4=0$

Sol^n: We know that regression lines passes through $\bar{x}, \bar{y}$.

$\therefore$ $2\bar{x}+3\bar{y}+1=0$

$\bar{x}+6\bar{y}-4=0$

Solving, we obtain $\bar{x}=-2, \bar{y}=1$

Now, to calculate 'r' we must identify the regression lines of y on x and x on y so that we can write the regression coefficients.

Suppose,

$2x+3y+1=0$ as $3y=-2x-1$ or

$$y=-\frac{2}{3}x-\frac{1}{3}$$

and $x+6y-4=0$ is written as $x=-6y+4$

the regression coefficients are $-\frac{2}{3}$ and -6.

$$\therefore r=\sqrt{\left(-\frac{2}{3}\right)\times(-6)}=\sqrt{4}=\pm 2$$

This cannot be considered as r lies between $-1 \le r \le 1$.

So, we have to change the given equations into other possible form.

$2x+3y+1=0$ is rewritten as $2x=-3y-1$

or $x=-\frac{3}{2}y-\frac{1}{2}$

and $x+6y-4=0$ is rewritten as

$$6y=-x+4 \text{ or } y=-\frac{x}{6}+\frac{2}{3}$$

Now, the regression coefficients are $-\frac{3}{2}$ & $-\frac{1}{6}$

so that, $r=\sqrt{\left(-\frac{3}{2}\right)\times\left(-\frac{1}{6}\right)}$

$$=\sqrt{\frac{1}{4}}=\pm\frac{1}{2}$$

The sign of r must be negative as both regression coefficients are negative.

$$r=-0.5$$

Thus, $\bar{x}=-2$, $\bar{y}=1$, $r=-0.5$

8a) If θ is the acute angle between the lines of regression, then show that $\tan\theta=\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\left[\frac{1-r^2}{r}\right]$

Explain the significance when $r=0$ and $r=\pm 1$.

Sol^n: As θ is the acute angle between the lines say, $y=m_1x+c_1$ and $y=m_2x+c_2$ then,

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$$

The lines of regression are,

$$y-\bar{y}=r\frac{\sigma_y}{\sigma_x}(x-\bar{x}) \longrightarrow (1)$$

and $$x-\bar{x}=r\frac{\sigma_x}{\sigma_y}(y-\bar{y}) \longrightarrow (2)$$

Equation (2) is rewritten as,

$$y-\bar{y} = \frac{\sigma_y}{r\sigma_y}(x-\bar{x}) \longrightarrow (3)$$

Slopes of eqns (1) & (3) are given by,

$$m_1 = \frac{r\sigma_y}{\sigma_x} \quad \text{and} \quad m_2 = \frac{\sigma_y}{r\sigma_x}$$

Substituting these in the formula for $\tan\theta$,

$$\tan\theta = \frac{\dfrac{\sigma_y}{r\sigma_x} - \dfrac{r\sigma_y}{\sigma_x}}{1 + r\dfrac{\sigma_y}{\sigma_x}\cdot\dfrac{\sigma_y}{r\sigma_x}}$$

$$= \frac{\dfrac{\sigma_y}{\sigma_x}\left(\dfrac{1}{r} - r\right)}{1 + \dfrac{\sigma_y^2}{\sigma_x^2}} = \frac{\dfrac{\sigma_y}{\sigma_x}\left(\dfrac{1-r^2}{r}\right)}{\dfrac{\sigma_x^2+\sigma_y^2}{\sigma_x^2}}$$

$$= \frac{\sigma_y}{\sigma_x}\left(\frac{1-r^2}{r}\right) \times \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2}$$

$$\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\left(\frac{1-r^2}{r}\right)$$

If, $r = \pm 1$, $\tan\theta = 0$. $\theta = 0$, implies that the two regression lines coincide & hence the variables are perfectly correlated. Also, if $r = 0$, $\tan\theta = \infty$ or $\theta = \frac{\pi}{2}$. This implies that the lines are perpendicular and hence the variables are uncorrelated.

8b. The following table gives the heights of fathers (x) and sons (y) :

x	65	66	67	67	68	69	70	72
y	67	68	65	68	72	72	69	71

Calculate the coefficient of correlation and lines of regression.

Sol^n: Here $n = 8$ and we prepare the following table.

x	y	$z = x - y$	x^2	y^2	z^2
65	67	-2	4225	4489	4
66	68	-2	4356	4624	4
67	65	2	4489	4225	4
67	68	-1	4489	4624	1
68	72	-4	4624	5184	16
69	72	-3	4761	5184	9
70	69	1	4900	4761	1
72	71	1	5184	5041	1
$\Sigma x = 544$	$\Sigma y = 552$	$\Sigma z = -8$	$\Sigma x^2 = 37028$	$\Sigma y^2 = 38132$	$\Sigma z^2 = 40$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{544}{8} = 68 \qquad \bar{z} = \frac{\Sigma z}{n} = \frac{-8}{8} = -1$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$$

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{37028}{8} - (68)^2$$

$$= 4{,}628.5 - 4624$$

$$= \underline{4.5}$$

$$\therefore \sigma_x = \sqrt{4.5} = \underline{2}.1213$$

$$\sigma_y^2 = \frac{\Sigma y^2}{n} - (\bar{y})^2 = \frac{38132}{8} - (69)^2$$

$$= 4766.5 - 4761$$

$$= 5.5$$

$$\therefore \sigma_y = \sqrt{5.5} = \underline{2}.3452$$

$$\sigma_z^2 = \frac{\Sigma z^2}{n} - (\bar{z})^2 = \frac{40}{8} - (-1)^2 = 5-1 = 4$$

$$\sigma_z = \sqrt{4} = 2$$

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \sigma_y} = \frac{4.5 + 5.5 - 4}{2 \times 2.1213 \times 2.3452}$$

$$\boxed{\therefore r = \frac{6}{9.9497} = 0.60303}$$

The regression lines are

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 69 = 0.60303 \times \frac{2.3452}{2.1213}(x - 68)$$

$$y - 69 = 0.6667(x - 68) = 0.6667x - 45.3356$$

$$\boxed{y = 0.6667x + 23.6644}$$

Also, $x - \bar{x} = r\dfrac{\sigma_x}{\sigma_y}(y - \bar{y})$

$$x - 68 = 0.60303 \times \frac{2.1213}{2.3452} (y-69)$$

$$x - 68 = 0.5455 (y - 69)$$

$$\Rightarrow x - 68 = 0.5455y - 37.6395$$

$$\Rightarrow \boxed{x = 0.5455y + 30.3605}$$

These are the lines of regression.

8a) A simply supported beam carries a concentrated load at P at its midpoint. Corresponding to various values of P, the maximum deflection of Y is measured and is given below:

P	100	120	140	160	180	200
Y	0.45	0.55	0.60	0.70	0.80	0.85

Find a best fitting straight line in the form $Y = a + bP$ to the above data & hence estimate Y when $P = 150$.

Sol^n: The normal equations associated with $Y = a + bP$ are,

$$\Sigma Y = na + b\Sigma P$$

$$\Sigma PY = a\Sigma P + b\Sigma P^2 \qquad \text{Also, } n = 6$$

The relevant table is as follows,

Scanned with CamScanner

P	y	Py	P^2
100	0.45	45	10000
120	0.55	66	14400
140	0.60	84	19600
160	0.70	112	25600
180	0.80	144	32400
200	0.85	170	40000
$\Sigma P = 900$	$\Sigma y = 3.95$	$\Sigma Py = 621$	$\Sigma P^2 = 142000$

The normal equation becomes,

$$6a + 900b = 3.95$$

$$900a + 142000b = 621$$

On solving,

$a = 0.047619047$, $b = 0.004071429$

$a \simeq 0.0476$ $\quad \approx 0.0041$

Thus, the required law is

$$y = 0.0476 + 0.0041P$$

Also, when $P = 150$, $y = 0.6626 \approx 0.66$

MODULE-5

9a) The joint probability distribution of discrete random variables X and Y is given as:

X \ Y	1	3	6
1	1/9	1/6	1/18
3	1/6	1/4	1/12
6	1/18	1/12	1/36

Determine (i) marginal distribution of X and Y
(ii) Are X and Y statistically independent?

Solⁿ: The marginal distributions of X and Y are,

x_i	1	3	6
$f(x_i)$	1/3	1/2	3/18

y_j	1	3	6
$g(y_j)$	1/3	1/2	3/18

$P(X=x_1, Y=y_1) = \frac{1}{9}$.

$J_{ij} = J_{11} = \frac{1}{9}$. Also, $P(X=x_i, Y=y_j) = J_{ij}$

$\therefore$ X and Y are statistically independent.

9b) A sample of 100 students is taken from a large population. The mean height of the students in this sample is 160 cm. Can it be reasonably regarded that in the population the mean height is 165 cm and the standard deviation is 10cm at 5% level of significance?

Solⁿ: Here, $N = 100$ (Population size)

$\bar{x} = 160$ cm (Sample Mean)

$\mu = 165$ cm (Population Mean)

$\sigma = 10$ cm (Standard Deviation)

H_0: The sample considered is a true random sample from the population.

$$z = \left|\frac{\bar{x} - \mu}{\sigma/\sqrt{N}}\right| = \left|\frac{160 - 165}{10/\sqrt{100}}\right| = \left|\frac{-5}{10/10}\right|$$

$$= |5|$$

Thus, here the z-score is outside the interval (-1.96, 1.96). As such, on the basis of the two-tailed test, we reject the hypothesis H_0 at 5% level of significance. Thus, the sample considered cannot be reasonably regarded as a truly random sample.

(c) A random sample of 10 boys had the following I.Q.: 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Does this data support the assumption of a population mean I.Q. of 100 at 5% level of significance? ($t_{0.05}$ for 9 d.f. is 2.262).

Solⁿ: Given I.Q. of boys

$x : 70, 120, 110, 101, 88, 83, 95, 98, 107, 100$

$\therefore n = 10$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{1}{10}(70+120+110+101+88+83+95+98+107+100)$$

$$= \frac{972}{10} = 97.2$$

$$s^2 = \frac{1}{n-1}\Sigma(x-\bar{x})^2$$

$$= \frac{1}{10-1}\left[(70-97.2)^2+(120-97.2)^2+(110-97.2)^2+(101-97.2)^2+(88-97.2)^2+(83-97.2)^2+(95-97.2)^2+(98-97.2)^2+(107-97.2)^2+(100-97.2)^2\right]$$

$$= \frac{1}{9}\times\left[1833.6\right] = 203.7333$$

$\therefore s = 14.2735$

Given the mean of population $\mu = 100$

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$t = \frac{97.2 - 100}{\left(\frac{14.2735}{\sqrt{10}}\right)} = \frac{-2.8}{4.5136}$$

$$t = |-0.6203| < 2.262$$

We note that this z-score is less than $t_c = 2.262$

We note that the data supports the assumption

We conclude that the data supports the assumption of a population mean I.Q.

10a) Explain the terms: (i) Null hypothesis (ii) Confidence intervals (iii) Type-I and Type-II errors.

Sol^n: Null hypothesis: For reaching statistical decisions, we start with some assumptions or guesses about the populations involved. Such assumptions / guesses, which may or may not be true, are called statistical hypothesis.

The statistical hypothesis formulated primarily to see whether it can be rejected is called null hypothesis.

For example, suppose we wish to show that a certain coin is biased. For this purpose, suppose we make the hypothesis that the coin is fair and proceed to find whether the hypothesis can be rejected. This hypothesis is a null hypothesis.

Confidence Interval: A confidence interval, in statistics, refers to the probability that a population parameter will fall between two set values for a certain proportion of times. Confidence intervals measure the degree of uncertainity or certainity in a sampling method. A confidence interval can take any number of probabilities, with the most common being a 95% or 99% confidence level.

Type I error: Rejecting the null hypothesis when it has to be accepted is called Type-I error.

Type II error: Accepting the null hypothesis when it has to be rejected is called Type-II error.

Scanned with CamScanner

10 b. A stenographer claims that she can type at the rate of 120 words per minute. Can we reject her claim on the basis of 100 trails in which she demonstrates a mean of 116 words with a standard deviation of 15 words? Use 5% level of significance.

<u>Sol</u>^{n}: Here, $n = 100$, $\mu = 120$, $\bar{x} = 116$, $\sigma = 15$

$$\text{Difference} = \bar{x} - \mu = 116 - 120 = -4$$

Null hypothesis : $H_0 : \mu = 120$ words

$$z = \frac{\text{Difference}}{\sigma/\sqrt{n}} = \frac{-4}{15/100} = -\frac{4}{15} \times 100$$

$$z = -26.67 < -1.96$$

$$z = 26.67 > 1.96 = z_{0.05}$$

Hence, H_0 is rejected.

(c) Four coins are tossed 100 times and the following results were obtained :

No. of heads	0	1	2	3	4
Frequency	5	29	36	25	5

Fit a binomial distribution for the data and test the goodness of fit ($\chi^2_{0.05}$ for 4 d.f. is 9.49)

Solⁿ: Let x denote the number of heads and f the corresponding frequency. Since the data is in the form of a frequency distribution we calculate the mean,

$$\text{Mean } (\mu) = \frac{\Sigma fx}{\Sigma f} = \frac{0+29+72+52+20}{100}$$

$$= \frac{196}{100} = 1.96$$

Also, $n = 4$. $\mu = np$.

$4p = 1.96 \Rightarrow p = 0.49$

$\therefore q = 1-p = 1-0.49 = 0.51$

Binomial distribution is given by,

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$= {}^4C_x (0.49)^x (0.51)^{4-x}$$

Since, 4 coins were tossed 100 times, expected frequencies are obtained from

$$F(x) = 100\,P(x) = 100\,{}^4C_x (0.49)^x (0.51)^{4-x}$$

where $x = 0, 1, 2, 3, 4.$

$$F(x=0) = 100\,(0.51)^4 = 6.75 \approx 7$$

$$F(x=1) = 100.\,{}^4C_1 (0.49)(0.51)^3 =$$

$$= 400\,(0.49)(0.51)^3 = 25.999$$

$$\simeq 26$$

$$F(x=2) = 100\,{}^4C_2 (0.49)^2 (0.51)^2$$

$$= 600 \times 0.2401 \times 0.2601$$

$$= 37.47 \simeq 37$$

$$F(x=3) = 100\,{}^4C_3 (0.49)^3 (0.51)$$

$$= 400 \times (0.117649)(0.51)$$

$$= 24.000396$$

$$\simeq 24$$

$$F(x=4) = 100\,{}^4C_4 (0.49)^4 (0.51)^0$$

$$= 100 \times 0.05765$$

$$\simeq 5.764801$$

$$\simeq 6$$

Thus, the required theoretical frequencies are 7, 26, 37, 24, 6.

O_i	5	29	36	25	5
E_i	7	26	37	24	6

$$\chi^2 = \frac{\Sigma (O_i - E_i)^2}{E_i}$$

$$= \frac{(5-7)^2}{7} + \frac{(29-26)^2}{26} + \frac{(36-37)^2}{37} + \frac{(25-24)^2}{24} + \frac{(5-6)^2}{6}$$

$$= \frac{4}{7} + \frac{9}{26} + \frac{1}{37} + \frac{1}{24} + \frac{1}{6} = 1.15$$

$$\chi^2 = 1.15 < \chi^2_{0.05} = 9.49$$

Thus, the hypothesis that the fitness is good can be accepted.