KLS's

Vishwanathrao Deshpande Institute of Technology, Haliyal

MODEL QUESTION PAPER -II

Department:Mathematics **Time: 3 Hours**

Semester / Division: IV Sem **Date:24-06-2021**

Subject with Sub. Code: Complex Analysis ,Probability distribution and statistical methods(18MAT41)

Note: Answer any **ONE** full question from each part; each full question carries 25 marks.

		PART-A	
1	a)	Derive Cauchy-Riemann equations in cartesian form.	**(6M)**
	b)	Determine analytic function f(z) whose imaginary part $\left(r-\frac{k^2}{r}\right)sin\theta\,,r\neq 0$ & hence find real part of $f(Z)$and Prove that it is harmonic.	**(6M)**
	c)	If f(z) is analytic S.T $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\lvert f(z)\rvert^2=4\lvert f'(z)\rvert^2$	**(7M)**
		OR	
2	a)	Derive Cauchy-Riemann equations in polar form.	**(6M)**
	b)	Show that $u=\left(r+\frac{1}{r}\right)\cos\theta$ is harmonic .Find its harmonic conjugate and also the corresponding analytic function.	**(6M)**
	c)	If f(z) is regular function of z than S.T $\left\{\frac{\partial}{\partial x}\lvert f(z)\rvert\right\}^2+\left\{\frac{\partial}{\partial y}\lvert f(z)\rvert\right\}^2=\lvert f'(z)\rvert^2$.	**(7M)**
		MODULE 2	
3	a)	Discuss the conformal transformations $w=z+\frac{1}{z}$	**(6M)**
	b)	Evaluate $\int_c \frac{e^z}{z+i\pi}dz$ a) $\lvert z\rvert=2\pi$ b) $\lvert z\rvert=\frac{\pi}{2}$ c) $\lvert z-1\rvert=1$	**(6M)**
	c)	Find the Bilinear Transformation which maps the points $z=\infty, i, 0$ into $W=-1,-i,1$.Also find the fixed point of the transformation.	**(7M)**
		OR	
4	a)	Evaluate $\int_c \frac{dz}{z^2-4}$ over the following curves i)c: $\lvert z\rvert$=3 ii) c: $\lvert z+2\rvert=1$	**(6M)**
	b)	Discuss the conformal transformations $w=z^2$	**(6M)**
	C)	Find the bilinear transformation which map the points z= 0, 1, ∞ into the points $w=-5,-1,3$ respectively	**(7M)**
		MODULE 3	
5	a)	Find the constant k such that $f(x)=\{Kx^2, 0<x<3\}$ 0, else where (i) P$(1<x<2)$ (ii) P(x ≤ 1)	**(6M)**
	b)	In a certain factory turning out razor blades there is a small probability of 0.002 for any blade to be defective. The blades are supplied in a packets of 10. Use Poisson distribution to calculate the approximate number of packets containing i) no defective ii) one defective iii) two defective blades in a consignment of 10,000 packets	**(6M)**
	c)	The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find a probability that a random call made from this booth i)ends less than 5 minutes ii) between 5 and 10 minutes.	**(7M)**
		OR	
6	a)	A random variable X has the following probability function for various values of x (table below) Find i) the value of K $ii)x<1$ $iii)\ x\geq -1$	**(6M)**
	b)	The number of telephone lines busy at an instant of time is a binomial variate with Probability 0.1 that a line is busy .If 10 lines are chosen at random ,what is the Probability that i) no line is busy ii)all lines are busy iii)at least one line is busy iv) at most 2 lines are busy.	**(6M)**

X	-2	-1	0	1	2	3
P(X)	0.1	K	0.2	2K	0.3	K

c) In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D. of 60 hours. If a firm purchases 2500 bulbs find the number of bulbs that are likely to last fori) more than 2100 hours ii) less than 1950 hours iii) between 1900 to 2100 hours **(7M)**

MODULE 4

7 a) Ten students got the following percentage of marks in two subjects x and y. Compute their rank Correlation coefficient. **(6M)**

Marks in X	78	36	98	25	75	82	90	62	65	39
Marks in Y	84	51	91	60	68	62	86	58	53	47

b) An Experiment on lifetime t of cutting tool at different cutting speeds v(units) are given below .Fit a relation of the form $v = at^b$ **(6M)**

V	350	400	500	600
t	61	26	7	2.6

c) The two regression equations of the Variable x and y are $x = 19.13 - 0.87y$ and $y = 11.64 - 0.50x$.Find the
i)Mean of x's ii)Mean of y's and iii) the correlation co-efficient between x and y. **(7M)**

OR

8 a) Fit a parabola $y = ax^2 + bx + c$ for the data **(6M)**

x	0	1	2	3	4
y	1	1.8	1.3	2.5	2.3

b) Obtain the lines of regression for the following values of x and y and hence find the Coefficient Correlation for the data. **(6M)**

x	1	2	3	4	5	6	7
y	9	8	10	12	11	13	14

c) Show that $tan\theta = \left(\frac{1-r^2}{r}\right)\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$ where θ is the angle between the lines of regression **(7M)**

MODULE 5

9 a) Determine i)The marginal distribution of X &Y. ii) Covariance between the discrete random variable X and Y of the joint probability distribution. **(6M)**

X \ Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

b) A certain stimulus administrated to each of the 12 patients resulted in the following change in blood pressure 5 , 2 , 8 , -1 , 3, 0 , 6 , -2 , 1 , 5 , 0 , 4 can it be concluded that the stimulus will increase the blood pressure ($t_{0.05}$ for 11 df = 2.201) **(6M)**

c) The mean of two large samples of 1000 and 2000 members are 168.75 cms and 170 cms respectively. Can the samples be regarded as drawn from the same population of standard deviation 6.25 cms? **(7M)**

OR

10 a) Explain the terms i) Null hypothesis, ii)Type-1, Type-2 Error iii) Confidence limit **(6M)**

b) Five dice were thrown 96 times and the numbers 1,2, or 3 appearing on the face of the dice follows the frequency distribution as below . Test the hypothesis that the data follows a binomial distribution ($\chi^2_{0.05}$ = 11.07 for 5 d.f) **(6M)**

No.of dice showing 1,2 or 3	5	4	3	2	1	0
Frequency	7	19	35	24	8	3

c) In a locality of 18000 families a sample of 840 families was selected at random. Of these 840 families, 206 families were found to have monthly income of Rs.2500 or less. It was desired to estimate how many of the 18,000 families have monthly income of Rs. 2500 or less. Within what limits would you place your estimate. **(7M)**

FOURTH SEMESTER B.E. Examination

MODEL QUESTION PAPER

Complex Analysis, Probability Distribution & Statistical method (18MAT41)

DETAILED SOLUTION

Module 1

1a Let $f(z)$ be analytic at a $z = x+iy$. By the definition of analytic function $f'(z)$ exist and is unique.

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

In cartesian form $f(z) = u(x,y) + iv(x,y)$ and δz be the increment in z corresponding to the increased $\delta x, \delta y$ in x, y

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y] + iv[\cancel{x}\, x+\delta x, y+\delta y] - [u(x,y) + iv(x,y)]}{\delta z}$$

$$\lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) - u(x,y)]}{\delta z} + i \lim_{\delta z \to 0} \frac{[v(x+\delta x, y+\delta y)] - v(x,y)}{\delta z}$$

$\because z = x+iy$; $\delta z = (z+\delta z) - z = \delta x + i\delta y$

Case i:- Let $\delta y = 0$ $\therefore \delta z = \delta x$, $\delta z \to 0$

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{------ (1)}$$

Case ii:- Let $\delta x = 0$ $\therefore \delta z \to 0 \Rightarrow i\delta y = 0$ or $\delta y \Rightarrow 0$

$$f'(z) = \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{---- (2)}$$

Equating eq (1) & (2)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

1.b. $v = (r - k^2/r)\sin\theta$

$v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta$; $v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$

Consider,

$f(z) = u + iv$

$f'(z) = e^{-i\theta}[u_r + iv_r] = e^{-i\theta}\left[\frac{1}{r}v_\theta + iv_r\right]$

$= e^{-i\theta}\left[\left(1 - \frac{k^2}{r^2}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right]$

$= e^{-i\theta}\left[(\cos\theta + i\sin\theta) - \frac{k^2}{r^2}(\cos\theta - i\sin\theta)\right]$

$= e^{-i\theta}\left[e^{i\theta} - \frac{k^2}{r^2}\cdot e^{-i\theta}\right] \Rightarrow 1 - \frac{k^2}{(re^{i\theta})^2} = 1 - k^2/z^2$

$f(z) = z + \frac{k^2}{z} + c$ putting $z = re^{i\theta}$

$f(z) = u + iv = re^{i\theta} + k^2/re^{i\theta}$

$= r[\cos\theta + i\sin\theta] + \frac{k^2}{r}[\cos\theta - i\sin\theta]$

$= \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$

$\therefore u = \left(r + \frac{k^2}{r}\right)\cos\theta$

we have to show that u is harmonic.

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

$u = [r + k^2/r]\cos\theta$

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{2k^2}{r^3}\cos\theta + \frac{1}{r}\cos\theta - \frac{k^2}{r^3}\cos\theta - \frac{1}{r}\cos\theta - \frac{k^2}{r^3}\cos\theta = 0$

1.c. Let, $f(z) = u + iv$

$|f(z)| = \sqrt{u^2 + v^2} \Rightarrow |f(z)|^2 = u^2 + v^2 = \phi$

Consider,

$\phi = u^2 + v^2$ Diff Partially w.r.t 'x'

$\phi_x = 2uu_x + 2vv_x = 2[uu_x + vv_x]$

Diff again

$\phi_{xx} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2]$ ------ (1)

lly $\phi_{yy} = 2[uu_{yy} + u_y^2 + vv_{yy} + v_y^2]$ ---- (2)

Adding eq^n (1) & (2)

$\phi_{xx} + \phi_{yy} = 2[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2]$

$\because f(z)$ is analytic $u_{xx}+u_{yy}=0$; $v_{xx}+v_{yy}=0$

$$\therefore \phi_{xx}+\phi_{yy}=2[u_x^2+v_x^2+(-v_x)^2+(u_x)^2]$$
$$=2[2(u_x^2+v_x^2)]=4[u_x^2+v_x^2]$$

$$\because f'(z)=u_x+iv_x \Rightarrow |f'(z)|^2=u_x^2+v_x^2$$

$$\therefore \phi_{xx}+\phi_{yy}=4|f'(z)|^2.$$

2a. Let $f(z)$ be analytic at the point $z=re^{i\theta}$. By the definition $f'(z)=\lim\limits_{\delta z\to 0}\dfrac{f(z+\delta z)-f(z)}{\delta z}$ exist and is unique.

In polar form $f(z)=u(r,\theta)+iv(r,\theta)$

let δz be the increment in z according to the increment $\delta r, \delta\theta$ in r,θ.

$$f'(z)=\lim_{\delta z\to 0}\frac{[u(r+\delta r,\theta+\delta\theta]+iv(r+\delta r,\theta+\delta\theta)]-[u(r,\theta)+iv(r,\theta)]}{\delta z} \quad ----(1)$$

$$=\lim_{\delta z\to 0}\frac{u[r+\delta r,\theta+\delta\theta]-u(r,\theta)}{\delta z}+i\lim_{\delta z\to 0}\frac{v(r+\delta r,\theta+\delta\theta)}{\delta z} \quad --(2)$$

$\because z=re^{i\theta}$

$$\delta z=\frac{\partial z}{\partial r}\delta r+\frac{\partial z}{\partial\theta}\delta\theta=e^{i\theta}\delta r+ire^{i\theta}\delta\theta$$

$\because \delta z\to 0$ there are two possiblities.

<u>Case i</u>:- Let $\delta\theta=0$ $\quad \because \delta z=e^{i\theta}\cdot\delta r.$

$\therefore \delta z\to 0 \Longrightarrow \delta r\to 0$ from eqn (2)

$$f'(z)=\lim_{\delta r\to 0}\frac{u(r+\delta r,\theta)-u(r,\theta)}{e^{i\theta}\cdot\delta r}+i\lim_{\delta r\to 0}\frac{v(r+\delta r,\theta)-v(r,\theta)}{e^{i\theta}\cdot\delta r}.$$

$$=e^{-i\theta}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right] \quad -----(3)$$

<u>Case ii</u>:- Let $\delta r\to 0$ $\quad \therefore \delta z=ire^{i\theta}\cdot\delta\theta$, $\delta z\to 0\Rightarrow\delta\theta=0$

from eqn (2)

$$f'(z)=\lim_{\delta\theta\to 0}\frac{u(r,\theta+\delta\theta)-u(r,\theta)}{ire^{i\theta}\cdot\delta\theta}+i\lim_{\delta\theta\to 0}\frac{v(r,\theta+\delta\theta)-v(r,\theta)}{ire^{i\theta}\cdot\delta r}$$

$$=e^{-i\theta}\left[\frac{-i}{r}\frac{\partial u}{\partial\theta}+\frac{1}{r}\cdot\frac{\partial v}{\partial\theta}\right] \quad -----(4)$$

Equating the R.H.S of equations (3) & (4)

$$\frac{\partial u}{\partial r}=\frac{1}{r}\,\partial v/\partial\theta \;;\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}.$$

2b] $u=(r+1/r)\cos\theta$

we shall show that.

$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$ ------ (1)

$u_r=(1-1/r^2)\cos\theta$; $u_{rr}=2/r^3\cos\theta$

$u_\theta=-(r+1/r)\sin\theta$; $u_{\theta\theta}=-(r+1/r)\cos\theta$

LHS of eqn (1) now becomes,

$\frac{2}{r^3}\cos\theta+\frac{1}{r}\cos\theta-\frac{1}{r^3}\cos\theta-\frac{1}{r}\cos\theta-\frac{1}{r^3}\cos\theta=0$

Thus the given u is harmonic.

To find v, let us consider C-R equation in the polar form

$ru_r=v_\theta$; $rv_r=-u_\theta$

$v_\theta=(r-1/r)\cos\theta$; $v_\theta=(1+\frac{1}{r^2})\sin\theta$

$v=\int(r-1/r)\cos\theta\,d\theta+f(r)$; $v=\int(1+1/r^2)\sin\theta\,dr+g(\theta)$

$v=(r-1/r)\sin\theta+f(r)$; $v=(r-1/r)\sin\theta+g(\theta)$

Comparing, we must have $f(r)=0$ & $g(\theta)=0$

Thus the required harmonic conjugate

$v=(r-1/r)\sin\theta$ Also we have,

$f(z)=u+iv$ $\Rightarrow$ $f(z)=(r+1/r)\cos\theta+i(r-1/r)\sin\theta$

Thus by putting $r=z$ & $\theta=0$

$f(z)=z+1/z.$

2c] Let, $f(z)=u+iv$

$f'(z)=u_x+iv_x \Rightarrow |f'(z)|^2=u_x^2+v_x^2$ ------ (1)

let,

$\phi=|f(z)|=\sqrt{u^2+v^2}$

Consider,

$\phi^2=u^2+v^2$ Diff w.r.t x partially

$2\phi\phi_x=2uu_x+2vv_x$

$\phi\phi_x=uu_x+vv_x$ ---- (1)

lly $\phi\phi_y=uu_y+vv_y$ ---- (2)

Squaring & adding eqn (1) & (2)

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$$\phi^2(\phi_x^2+\phi_y^2) = (uu_x+vv_x)^2+(uu_y+vv_y)^2$$

$$=(u^2u_x^2+v^2v_x^2+2uvu_xv_x)+(u^2u_y^2+v^2v_y^2+2uvu_yv_y)$$

$\because f(z)$ is analytic by C-R equations.

$$\phi^2(\phi_x^2+\phi_y^2) = (u^2u_x^2+v^2v_x^2+2uvu_xv_x)+(u^2v_x^2+v^2v_x^2-2uvv_xu_x)$$

$$= u^2[u_x^2+v_x^2]+v^2[u_x^2+v_x^2]$$

$$=(u^2+v^2)[u_x^2+v_x^2]$$

$$\phi^2(\phi_x^2+\phi_y^2) = \phi^2(u_x^2+v_x^2)$$

$$\phi_x^2+\phi_y^2 = u_x^2+v_x^2 \quad \text{ie} \left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2=|f'(z)|^2$$

Module 2.

3(a) Consider $W=z+1/z$ Putting $z=re^{i\theta}$

$$u+iv = re^{i\theta}+\frac{1}{re^{i\theta}} = r[\cos\theta+i\sin\theta]+\frac{1}{r}[\cos\theta-i\sin\theta]$$

$$=(r+1/r)\cos\theta+i(r-1/r)\sin\theta$$

$$u=(r+1/r)\cos\theta \quad ; \quad v=(r-1/r)\sin\theta \text{ ---- (1)}$$

we have eliminate r and θ.

To eliminate θ from (1)

$$\cos\theta=\frac{u}{r+1/r} \qquad \sin\theta=\frac{v}{r-1/r}.$$

$$\sin^2\theta+\cos^2\theta = \left[\frac{u^2}{(r+1/r)^2}+\frac{v^2}{(r-1/r)^2}\right]$$

$$=\frac{u^2}{(r+1/r)^2}+\frac{v^2}{(r-1/r)^2}=1 \text{ ----- (2)}$$

To eliminate 'r'

$$\frac{u}{\cos\theta}=[r+1/r] \qquad \frac{v}{\sin\theta}=(r-1/r)$$

Squaring and substracting

$$\frac{u^2}{\cos^2\theta}-\frac{v^2}{\sin^2\theta}=\left[r+\frac{1}{r}\right]^2-\left[r-\frac{1}{r}\right]^2=4 \text{ ----- (3)}$$

or

$$\frac{u^2}{(2\cos\theta)^2}-\frac{v^2}{(2\sin\theta)^2}=1.$$

Case i :- Let, $r =$ constant

eqⁿ (2) becomes.

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$$

which represents an ellipse with foci $(\pm\sqrt{A^2-B^2}, 0) = (\pm 2, 0)$

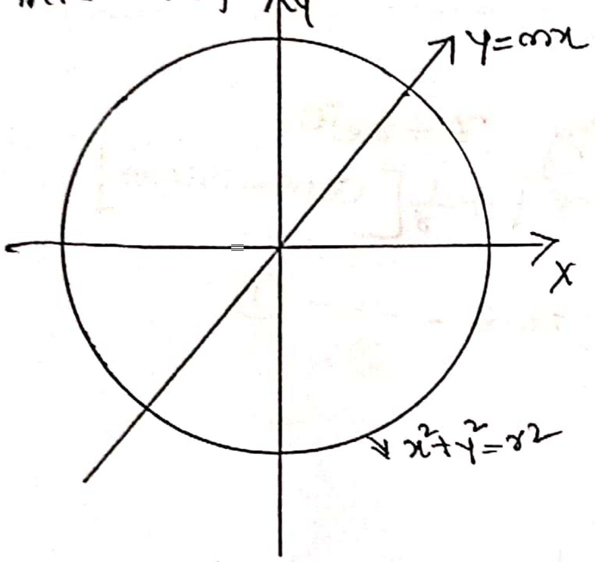
hence, $|z| = r =$ constant in the z-plane maps onto an ellipse in the W-plane with foci $(\pm 2a, 0)$

Case ii :- Let $\theta =$ constant.

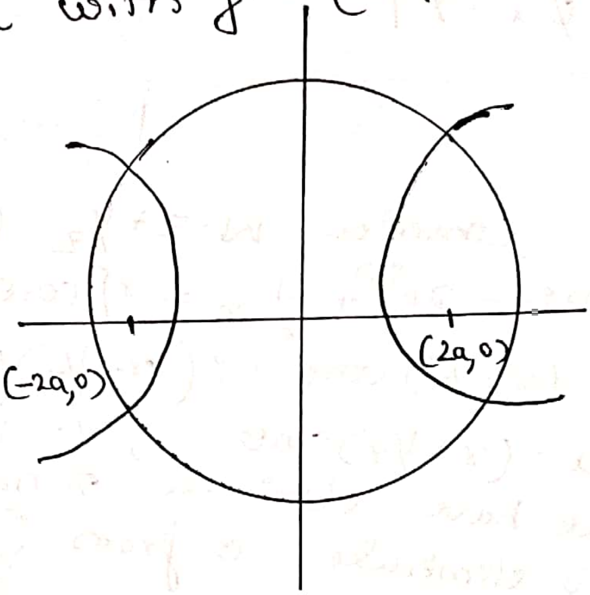
From eqⁿ (3)

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$$

This represents hyperbola with foci $(\pm\sqrt{A^2+B^2}, 0) = (\pm 2, 0)$

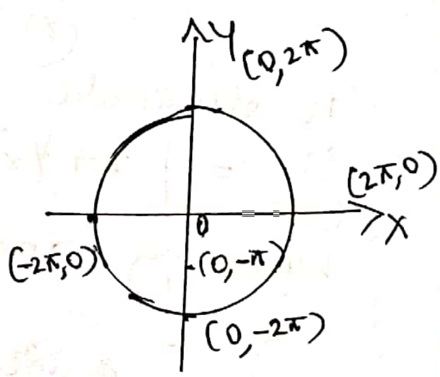


Z-plane.



3b] Given $\int_C \frac{f(z)}{z-a}\,dz = \int \frac{e^z}{z+i\pi}\,dz.$

$f(z) = e^z \qquad a = -i\pi$



i) $|z| = 2\pi$

$a = -i\pi$ is pt lies with the ⊙le

$|z| = 2\pi$

$\therefore$ By cauchys Integral formula.

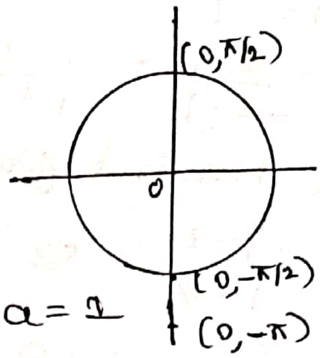
$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\int \frac{e^z}{z+i\pi}\,dz = 2\pi i f(-i\pi) = 2\pi i e^{-i\pi}$$

$$= 2\pi i[\cos\pi - i\sin\pi] = -2\pi i$$

ii] $|z| = \pi/2$ is a circle with centre at origin & radius $\pi/2$ (7)

The point $(0, -\pi)$ lies outside the circle



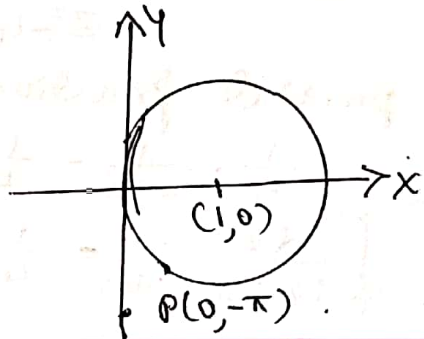
$$\therefore \int_C \frac{e^z}{z+i\pi}\, dz = 0$$

iii] $|z-1| = 1$ is a circle with centre at $z = a = 1$ & radius 1.

The point $P(0, -\pi)$ lies outside the circle

$|z-1| = 1$ and hence by Cauchy's theorem $\int_C \frac{e^z}{z+i\pi}\, dz = 0$

where, $C : |z-1| = 1$.



3c] Let, $w = \frac{az+b}{cz+d}$ be the required bilinear transformation

$z = \infty$, $w = -1$

$$w = \frac{z[(a+b/z)]}{z[(c+d/z)]} = \frac{a+b/z}{c+d/z}$$

$$\Rightarrow -1 = \frac{a+0}{c+0} \Rightarrow a + c = 0 \quad \text{------} (1)$$

$z = i$, $w = -i$; $-i = \frac{ai+b}{ci+d} \Rightarrow ai + b - c + di = 0$ ------ (2)

$z = 0$, $w = 1 \Rightarrow b - d = 0$ ------ (3)

Adding (1) & (2)

$(1+i)a + b + id = 0$ ------ (4)

Solving (3) & (4)

$$\left.\begin{array}{l} 0.a + 1.b - 1.d = 0 \\ (1+i)a + 1.b + i.d = 0 \end{array}\right\} \Rightarrow \frac{a}{i+1} = \frac{-b}{1+i} = \frac{d}{-(1+i)} = 1.$$

$$\frac{a}{1} = \frac{-b}{1} = \frac{-d}{1} = (1+i) = k.$$

$a = k$, $b = -k$, $d = -k$ or $a = 1$, $b = -1$, $d = -1$

From (i)

$C = -1$

$W = \frac{z-1}{-z-1} = \frac{1-z}{1+z}$.

The invariant points are obtained by putting $W = z$

$z = \frac{1-z}{1+z}$ or $z + z^2 = 1 - z$

$\Rightarrow z^2 + 2z - 1 = 0 \Rightarrow z = -1 \pm \sqrt{2}$

$\therefore$ The invariant or fixed points are $-1+\sqrt{2}, -1-\sqrt{2}$.

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4 a] Consider $\frac{1}{z^2-4} = \frac{1}{(z+2)(z-4)} = \frac{A}{z-2} + \frac{B}{z+4}$

partial fraction

$= \frac{1}{4} \cdot \frac{1}{z-2} - \frac{1}{4} \cdot \frac{1}{z+4}$

$\int_C \frac{dz}{(z-2)(z+4)} = \frac{1}{4} \int_C \frac{dz}{z-2} - \frac{1}{4} \int_C \frac{dz}{z-(-2)}$

i] $|z| = 3$, $z = a = 2$ and $z = a = -2$ both lie inside

$\therefore \int_C \frac{dz}{z^2-4} = \frac{2\pi i}{4} [f(z) - f(-z)] = 0$

ii] $|z+2| = 1$ is circle with centre $(-2,0)$ and radius 1. Let $A = (-2,0)$ and $P = (2,0)$

$AP = \sqrt{4} = 2 > 1$

$\therefore$ The point $z = a = 2$ lies outside the ok and $z = a = -2$ lies inside the ok

$\therefore \int_C \frac{dz}{z+2} = \int_C \frac{dz}{z-(-2)} = 2\pi i f(-2) = 2\pi i$

$\therefore \int_C \frac{dz}{z^2-4} = \frac{1}{4} \times 0 - \frac{1}{4} 2\pi i = \frac{-i\pi}{2}$.

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4b] Consider $W = z^2$

$u + iv = (x+iy)^2 = (x^2-y^2) + i(2xy)$

$u = x^2 - y^2 \qquad v = 2xy$

<u>Case i:-</u> Let, $x = c_1$ where c_1 is constants.

$u = c_1^2 - y^2$, $v = 2c_1 y \implies y = v/2c_1$

$\therefore u = c_1^2 - \left(v^2/4c_1^2\right)$ or $\frac{v^2}{4c_1^2} = c_1^2 - u$.

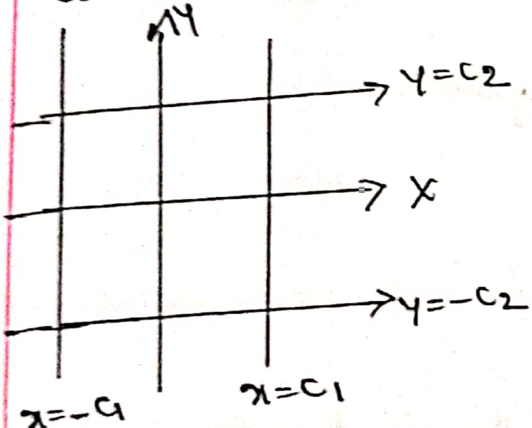
$v^2 = 4c_1^2(c_1^2 - u) = -4c_1^2(u - c_1^2)$

This is parabola in the W-plane symmetrical about the real axis with its vertex $(c_1^2, 0)$ & focus at the origin.

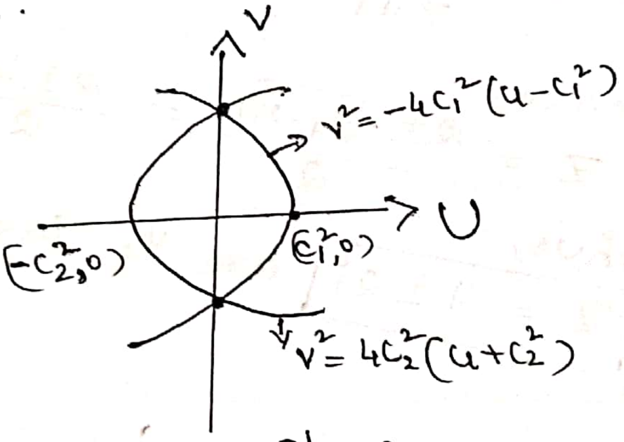
<u>Case ii:-</u> Let $y = c_2$, c_2 is constant.

$u = x^2 - c_2^2$, $v = 2xc_2$ or $v^2 = 4c_2^2(u + c_2^2)$

This is also a parabola in the W-plane symmetrical about the real axis whose vertex is at $(-c_2^2, 0)$ and Focus at the origin.



Z-plane



W = plane.

4c] Let, $W = \frac{az+b}{cz+d}$ be the required bilinear transformation

$z = 0,\ W = -5$; $-5 = b/d$.

ie $b = -5d$ ------ (1)

$z = 1,\ W = -1$; $-1 = \frac{a+b}{c+d}$

ie $a + b + c + d = 0$ ----- (2)

$z=\infty,\ w=3$; $w = \dfrac{z[a+(b/z)]}{z[c+(d/z)]} = \dfrac{a+(b/z)}{c+(d/z)}$

Now, consider,

$z=\infty,\ w=3$; $3 = \dfrac{a+0}{c+0}$

ie $a = 3c$ ------ (3)

Using (1) and (3) in (2) we get $4c-4d=0$

or $c=d$.

Choosing, $d=1$ we get $c=1,\ b=-5,\ a=3$

Thus the required bilinear transformation is

$w = \dfrac{3z-5}{z+1}$

The invariant points are obtained by taking $w=z$.

ie $z = \dfrac{3z-5}{z+1}$ or $z^2+z = 3z-5$ or $z^2-2z+5=0$

hence,

$z = \dfrac{2\pm\sqrt{4-20}}{2} = \dfrac{2\pm 4i}{2} = 1\pm 2i$

Thus, $\boxed{z = 1\pm 2i}$ are the invariant points.

Module 3.

5a]. By the condition of Probability density function

$f(x) \geq 0$ if $k>0$

Also, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

ie $\displaystyle\int_0^3 kx^2\,dx = 1$ or $9k=1 \Rightarrow k = 1/9$.

i] $P(1<x<2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{x^2}{9}\,dx = \left[\frac{x^3}{27}\right]_1^2$

$= \frac{7}{27}$

ii] $P(x\le 1) = \int_0^1 f(x)\,dx = \left[\frac{x^3}{27}\right]_0^1 = \frac{1}{27}$.

5b] Here,
p is the Probability of a defective blade $= \frac{1}{500} = 0.002$
In a packet of 10, the mean number of defective blade is,

mean $= m = np = 10\times 0.002 = 0.02$

poission distribution is

$P(x) = \frac{m^x \cdot e^{-m}}{x!} = \frac{e^{-0.02}(0.02)^x}{x!}$

Let,
$f(x) = 10{,}000\,P(x)$ Also, $e^{-0.02} \simeq 0.9802$

$f(x) = \frac{9802(0.02)^x}{x!}$

i] Probability of no defective $= f(0) = \frac{9802(0.02)^0}{0!} = 9802$

ii] Probability of one defective $= f(1) = \frac{9802(0.02)^1}{1!} = 196$

iii] Probability of two defective $= f(2) = \frac{9802(0.02)^2}{2!} \simeq 2$.

5c] we have,
The continuous probability distribution having the Probability density function $f(x)$ is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x>0 \\ 0 & \text{otherwise} \end{cases} \quad \text{where } \alpha>0.$$

here, $f(x) = \alpha e^{-\alpha x}$, $0<x<\infty$

Given

$1/\alpha = 5 \quad \therefore \alpha = 1/5$

i] $P(x<5) = \int_0^{\infty} \alpha e^{-\alpha x} dx = \int_0^{5} \frac{1}{5} \cdot e^{-x/5} dx = 0.6321$

ii] $P(5<x<10) = \int_0^{\infty} \alpha e^{-\alpha x} dx = \int_5^{10} \frac{1}{5} e^{-x/5} dx = -\left[e^{-x/5}\right]_5^{10} = 0.2325.$

64] we have,

$P(x_i) \geq 0$ and $\Sigma P(x) = 1$

X	-2	-1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	k

$0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$0.6 + 4k = 1$

$4k = 1 - 0.6 \Rightarrow 4k = 0.4$

$\boxed{k = 0.1}$

X	-2	-1	0	1	2	3
P(x)	0.1	0.1	0.2	0.2	0.3	0.1

Mean $= \Sigma x \cdot P(x)$

$= -2(0.1) + (-1)(0.1) + 0 + 1(0.2) + 2(0.3) + 3(0.1)$

$= -0.2 - 0.1 + 0.2 + 0.6 + 0.3$

Mean $(\mu) = 0.8$

Variance $= \Sigma (x-\mu)^2 P(x)$

$= (-2-0.8)^2 0.1 + (-1-0.8)^2 0.1 + 0 + (1-0.8)^2 0.2 + (2-0.8)^2 0.3$

$+ (3-0.8)^2 0.1$

$V = 2.16$

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6b] Let x denote the number of telephone line busy.
for this variate
we have,

$p = 0.1$; $q = 1-p = 0.9$; $n = 10$

$p(x) = nC_x p^x \cdot q^{n-x}$

$= 10C_x (0.1)^x (0.9)^{10-x}$

i] Probability that No line is busy

$p(x) = 10C_x (0.1)^x (0.9)^{10-x}$

$p(0) = 10C_0 (0.1)^0 (0.9)^{10-0} = 0.3487.$

ii] Probability that all lines are busy

$p(x) = 10C_x (0.1)^x (0.9)^{10-x}$

$p(10) = 10C_{10} (0.1)^{10} (0.9)^{10-10} = (0.1)^{10}.$

iii] Probability that atleast one line is busy

= 1 − Probability of no line is busy

$= 1 - p(0) = 1 - 0.3487 = 0.6513.$

iv] Probability that atmost 2 lines are busy.

$p(x \le 2) = p(0) + p(1) + p(2)$

$= 10C_0 (0.1)^0 (0.9)^{10} + 10C_1 (0.1)^1 (0.9)^9 + 10C_2 (0.1)^2 (0.9)^8$

$p(x \le 2) = 0.9298$

6c] By the data,

$\mu = 2000$, $\sigma = 60$

we have standard normal variate

$z = \frac{x-\mu}{\sigma} = \frac{x-2000}{60}$

i) To find $P(x > 2100)$

If $x = 2100$, $z = 100/60 \simeq 1.67$

$$P(x > 2100) = P(z > 1.67)$$
$$= P(z \geq 0) - P(0 < z < 1.67)$$
$$= 0.5 - \phi(1.67)$$
$$= 0.5 - 0.4525 = 0.0475.$$

$\therefore$ The number of bulbs that are likely to last for more than 2100 hours is.

$2500 \times 0.0475 = 118.75 \simeq 119.$

ii) To find $P(x < 1950)$

If $x = 1950$, $z = -5/6 = -0.83$

$$P(x < 1950) = P(z < -0.83)$$
$$= P(z > 0.83)$$
$$= P(z \geq 0) - P(0 < z < 0.83)$$
$$= 0.5 - \phi(0.83)$$
$$= 0.5 - 0.2967 = 0.2033.$$

$\therefore$ Number of Bulbs that are likely to last for less than 1950 hours is,

$2500 \times 0.2033 = 508.25 \simeq 508.$

iii) To find $P(1900 < x < 2100)$

If $x = 1900$, $z = -1.67$ and if $x = 2100$, $z = 1.67$

$$P(1900 < x < 2100) = P(-1.67 < z < 1.67)$$
$$= 2P(0 < z < 1.67)$$
$$= 2\phi(1.67) = 2 \times 0.4525 = 0.905$$

$\therefore$ The Number of bulbs that are likely to last between 1900 & 2100 hours

$= 2500 \times 0.905 = 2262.5 \simeq 2263.$

7a We prepare the table consisting of the given data along with the ranks assigned according to their order of the magnitude. In the subject x, 98 will be awarded rank 1, 90 as rank 2 and so on.

Marks in X	Rank(x)	Marks in Y	Rank (Y)	$d=(x-y)$	$d^2=(x-y)^2$
78	4	84	3	1	1
36	9	51	9	0	0
98	1	91	1	0	0
25	10	60	6	4	16
75	5	68	4	1	1
82	3	62	5	-2	4
90	2	86	2	0	0
62	7	58	7	0	0
65	6	53	8	-2	4
39	8	47	10	-2	4
					$\Sigma d^2 = 30$

we have

$$\rho = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$ and $n = 10$ for the given data.

$$\rho = 1 - \frac{6(30)}{10(10^2-1)} = 0.81818 \simeq 0.82$$

Thus,

$$\boxed{\rho = 0.82}$$

7b] Consider,

$$v = at^b$$

$$\log_e v = \log_e a + b \log_e t$$

Let,

$V = \log_e v$, $A = \log_e a$, $T = \log_e t$ and hence we have the equation

$$V = A + bT$$

The associated normal equations are as follows.

$$\Sigma V = nA + b\Sigma T$$

$$\Sigma VT = A\Sigma T + b\Sigma T^2 \quad (n=4).$$

v	t	$V = \log_e v$	$T = \log_e t$	VT	T^2
350	61	5.8579	4.1109	24.0812	16.8995
400	26	5.9915	3.2581	19.5209	10.6152
500	7	6.2146	1.9459	12.093	3.7865
600	2.6	6.3969	0.9555	6.1122	0.913
		$\Sigma V = 24.4609$	$\Sigma T = 10.2704$	$\Sigma VT = 61.8073$	$\Sigma T^2 = 32.2142$

The normal equations become.

$$4A + 10.2704\,b = 24.4609$$

$$10.2704\,A + 32.2142\,b = 61.8073$$

On solving,

we have,

$$A = 6.5539 \quad \text{and} \quad b = -0.1709$$

$$A = \log_e a \Rightarrow a = e^A = e^{6.5539} = 701.9766$$

Thus the required curve of fit is

$$v = (701.9766)\,t^{-0.1709}.$$

7c] we have to compute $\bar{x}, \bar{y}$ and r

w.k.T the regression lines passes through $(\bar{x}, \bar{y})$

$\bar{x} = 19.13 - 0.87y$ and $\bar{y} = 11.64 - 0.5x$ ------ (1)

That is,

$\bar{x} + 0.87\bar{y} = 19.13$ ------ (1)

$0.5\bar{x} + \bar{y} = 11.64$ ------ (2)

we have to simultaneously solve (1) and (2)

On solving we get,

$\bar{x} = 15.96$ and $\bar{y} = 3.67$.

w.k.T

$r = \sqrt{(\text{coeff of } x)(\text{coeff of } y)} = \sqrt{(-0.5)(-0.87)}$

$r = \pm 0.66$. Since both the regression coefficients are negative, we must take the sign of r to be negative.

Thus,

$\bar{x} = 15.96$, $\bar{y} = 3.67$, $r = -0.66$.

8a

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

The normal equations associated with $y = a + bx + cx^2$ are as follows.

$\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$ $(n = 5)$

x	y	xy	x^2y	x^2	x^3	x^4
0	1	0	0	0	0	0
1	1.8	1.8	1.8	1	1	1
2	1.3	2.6	5.2	4	8	16
3	2.5	7.5	22.5	9	27	81
4	6.3	25.2	100.8	16	64	256
$\Sigma x=10$	$\Sigma y=12.9$	$\Sigma xy=37.1$	$\Sigma x^2y=130.3$	$\Sigma x^2=30$	$\Sigma x^3=100$	$\Sigma x^4=354$.

The normal equations become

$5a+10b+30c=12.9$

$10a+30b+100c=37.1$

$30a+100b+354c=130.3$.

On solving we have,

$a=1.42$, $b=-1.07$, $c=0.55$

Thus the parabola of fit is,

$y=1.42-(1.07)x+0.55x^2$

8b] We first compute $\bar{x}, \bar{y}$ and denote $X=x-\bar{x}$, $Y=y-\bar{y}$ to form the table.

Here, $\bar{x}=4$, $\bar{y}=11$ $\therefore X=x-4$, $Y=y-11$

x	y	X	Y	XY	X^2	Y^2
1	9	-3	-2	6	9	4
2	8	-2	-3	6	4	9
3	10	-1	-1	1	1	1
4	12	0	1	0	0	1
5	11	1	0	0	1	0
6	13	2	2	4	4	4
7	14	3	3	9	9	9
				$\Sigma XY=26$	$\Sigma X^2=28$	$\Sigma Y^2=28$.

we shall consider regression lines in the form

$$Y = \frac{\Sigma XY}{\Sigma X^2} . X \quad \text{and} \quad X = \frac{\Sigma XY}{\Sigma Y^2} . Y$$

$$y - 11 = \frac{26}{28}(x-4) \quad ; \quad x - 4 = \frac{26}{28}(y-11)$$

$$y - 11 = 0.93(x-4) \quad ; \quad x - 4 = 0.93(y-11)$$

Thus,

$y = 0.93x + 7.28$ and $x = 0.93y - 6.23$.

These are the regression lines we compute r as the geometric mean of the regression coefficients.

$$r = \sqrt{(\text{Coeff of } x)(\text{Coeff of } y)} = \sqrt{(0.93)(0.93)} = 0.93.$$

$$\boxed{r = 0.93.}$$

8 c) W.K.T if θ is acute, the angle between the lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

we have the lines of regression

$$y - \bar{y} = r . \frac{\sigma y}{\sigma x}(x - \bar{x}) \text{ - - - - (1)} \quad \text{and} \quad x - \bar{x} = r . \frac{\sigma x}{\sigma y}(y - \bar{y})$$

we write the second of the equation

$$y - \bar{y} = \frac{\sigma y}{r \sigma x}(x - \bar{x}) \text{ - - - - (2)}$$

Slopes of 1 and 2 are respectively

$$m_1 = \frac{r\sigma y}{\sigma x} \quad \text{and} \quad m_2 = \frac{\sigma y}{r\sigma x}.$$

Substituting these in the formula for $\tan\theta$

$$\tan\theta = \frac{\frac{\sigma y}{r\sigma x} - \frac{r\sigma y}{\sigma x}}{1 + \frac{r\sigma y}{\sigma x} . \frac{\sigma y}{r\sigma x}} = \frac{\frac{\sigma y}{\sigma x}\left(\frac{1}{r} - r\right)}{1 + \frac{\sigma^2 y}{\sigma^2 x}}$$

$$= \frac{\frac{\sigma y}{\sigma x}\left(\frac{1 - r^2}{r}\right)}{\frac{\sigma^2 x + \sigma^2 y}{\sigma^2 x}} \quad \text{Thus,} \quad \tan\theta = \frac{\sigma x \sigma y}{\sigma^2 x + \sigma^2 y}\left(\frac{1 - r^2}{r}\right)$$

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9 a] Marginal distributions of X and Y is as follows.

x_i	2	4	6
$f(x_i)$	1/4	1/2	1/4

y_j	1	3	9
$g(y_j)$	1/2	1/3	1/6

$Cov(X,Y) = E(XY) - \mu_x \mu_y$

Let,

$E(X) = \sum_i x_i f(x_i)$

$= 2 \times 1/4 + 4 \times 1/2 + 6 \times 1/4 = 4$

$E(Y) = \sum_j y_j g(y_j)$

$= 1 \times 1/2 + 3 \times 1/3 + 9 \times 1/6 = 3$

$E(XY) = \sum_{i,j} x_i y_j J_{ij}$

$= (2)(1)(1/8) + (2)(3)(1/24) + (2)(9)(1/12) + (4)(1)(1/4) + (4)(3)(1/4)$
$+ (4)(9)(0) + (6)(1)(1/8) + (6)(3)(1/24) + (6)(9)(1/12)$

$E(XY) = 12.$

we have, from

i] $cov(X,Y) = 12 - (4)(3) = 0$

Thus

$cov(X,Y) = 0$

Correlation of X and $Y = \rho(X,Y) = \dfrac{cov(X,Y)}{\sigma_x \sigma_y}$

Thus, $\boxed{\rho(X,Y) = 0.}$

9b] Let,

$$\bar{x} = \frac{\Sigma x}{n} = \frac{31}{12} = 2.5833 \simeq 2.58$$

$$s^2 = \frac{1}{n-1} \Sigma (x-\bar{x})^2$$

$$s^2 = \frac{1}{11}\{(5-2.58)^2 + (2-2.58)^2 + (8-2.58)^2 + (-1-2.58)^2 + (3-2.58)^2 + (-2-2.58)^2 + (1-2.58)^2 + (5-2.58)^2 + (0-2.58)^2 + (6-2.58)^2 + (0-2.58)^2 + (4-2.58)^2\}$$

$$= 9.538$$

$$s = 3.088.$$

we have,

$$t = \frac{\bar{x}-\mu}{s}\sqrt{n}.$$

Let us suppose that the stimulus administration is not accompanied with increase in blood pressure,

we can take $\mu = 0$

$$t = \frac{2.5833-0}{3.088}\sqrt{12} = 2.8979 \simeq 2.9 > 2.201.$$

Hence the hypothesis is rejected at 5% level of significance we conclude with 95% confidence that the stimulus in general is accompanied with increase in blood pressure

---

7.9] Let,

$\bar{x}_1 = 168.75$; $\bar{x}_2 = 170$

$n_1 = 1000$; $n_2 = 2000$

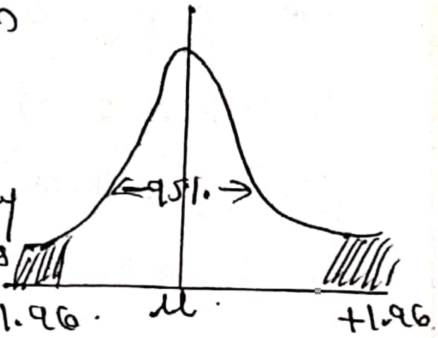
$$z = \frac{\bar{x}_2 - \bar{x}_1}{\sigma\sqrt{1/n_1 + 1/n_2}} = \frac{1.25}{6.25\sqrt{1/1000 + 1/2000}} = 5.16.$$

$z = 5.16$ is very much greater than $z_{0.05} = 1.96$ and also $z_{0.01} = 2.58$. Thus we say that the difference between the sample means is significant and we conclude that the samples cannot be regarded as drawn from the same populations.

10a] i] Null Hypothesis: The hypothesis formulated for the purpose of its rejection under the assumption than it is true is called the null hypothesis. and it is denoted by H_0. 22

ii] Confidence limits:-

In this normal probability Distribution curve the normal areas we find that 95% of the area lies between $z=-1.96$ and $z=+1.96$. So this area we can say that 95% of confidence limit that z lies between $(-1.96, +1.96)$.



iii] Type I & Type II Error:-

If a hypothesis is rejected while it should have been accepted it is known as Type I Error.

If a hypothesis is accepted while it should have been rejected is known as Type II Error.

10b] The data gives the observed frequencies and need to calculate the expected frequencies probability of single dice throwing 1, 2 or 3 is,

$p = 3/6 = 1/2 \quad \therefore q = 1 - p = 1/2$

The Binomial distribution of fit is

$N(q+p)^n = 96\,(1/2 + 1/2)^5$

The theoretical frequencies of getting 5, 4, 3, 2, 1, 0 successes with 5 dice are respectively the successive terms of the binomial expansion

$96 \times 1/2^5, \quad 96 \times 5C_1 \times 1/2^5, \; - - - - \; 96 \times 1/2^5$

or 3, 15, 30, 30, 15, 3

we have the table of observed & expected frequency.

Oi	7	19	35	24	8	3
Ei	3	15	30	30	15	3

the
ix
220

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

$$= \frac{16}{3} + \frac{16}{15} + \frac{25}{30} + \frac{36}{30} + \frac{49}{15} + \frac{0}{3}$$

$$= 11.7$$

$$\chi^2 = 11.7 > \chi^2_{0.05} = 11.07$$

Thus the hypothesis that the data follows a binomial distribution is rejected.

O C) Proportion of families having monthly income of Rs 2500 or less is given by

$p = 206/840 = 0.245$. Hence $q = 1 - p = 0.755$

S.E proportion $= \sqrt{pq/n} = \sqrt{(0.245 \times 0.755)/840}$

$= 0.015$

Probable limits of families having monthly income of Rs 2500 or less are $p \pm 2.58\sqrt{pq/n}$

That is,

$= 0.245 \pm (2.58)(0.015)$

$= 0.245 \pm 0.0387$

$= 0.2063$ and 0.2837

or 20.63 % and 28.37 %

Hence the Probable limits in respect of 18,000 families is given by $0.2063 \times 18{,}000$ and $0.2837 \times 18{,}000$ or 3713 and 5107

That is, 3713.4 and 5106.6

Thus we say that 3713 to 5107 families are likely to have monthly income of Rs 2500 or less.

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