# MODEL QUESTION PAPER
## Fourth Semester BE Degree (CBCS) Examination 2020-21

## Additional Mathematics-II [18MATDIP41]

**Time: 3 hours** **Max Marks: 100**

***Note: Answer any FIVE full questions, choosing ONE full question from each module.***

### Module - 1

| | | |
|---|---|---|
| **1 a)** | Find the rank of the following matrix by elementary row transformations $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$. | (6m) |
| **b)** | Solve the following system of equations by Gauss elimination method, $x + y + z = 9; x - 2y + 3z = 8; 2x + y - z = 3$. | (7m) |
| **c)** | Find the eigen values and the corresponding eigen vectors of $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$. | (7m) |
| | OR | |
| **2 a)** | Find the rank of the following matrix by elementary row transformations $A = \begin{bmatrix} 0 & 1 & -3 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$. | (6m) |
| **b)** | Solve the following system of equations by Gauss elimination method, $2x + 5y + 7z = 52; 2x + y - z = 0; x + y + z = 9$. | (7m) |
| **c)** | Find the eigen values and the corresponding eigen vectors of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. | (7m) |

### Module – 2

| | | |
|---|---|---|
| **3 a)** | The area of a circle(A) corresponding to diameter (D) is given below. Find the area corresponding to diameter 105 by using an appropriate interpolation.<br>D: 80, 85, 90, 95, 100<br>A: 5026, 5674, 6362, 7088, 7854 | (6m) |
| **b)** | Find a real root of $x^3 - 2x - 5 = 0$ correct to three decimal places by Newton Raphson method. | (7m) |
| **c)** | Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's $\frac{1}{3}^{rd}$ rule taking four equal strips. | (7m) |
| | OR | |
| **4 a)** | From the following table, find the number of students who have obtained less than 45 marks.<br>Marks: 30-40, 40-50, 50-60, 60-70, 70-80<br>Students: 31, 42, 51, 35, 31 | (6m) |
| **b)** | Find the fourth root of 12 correct to three decimal places by Regula -Falsi method | (7m) |
| **c)** | Evaluate $\int_4^{5.2} \log_e x \; dx$ taking 6 equal strips by using Weddle's rule. | (7m) |

Table for 3 a):

| D | 80 | 85 | 90 | 95 | 100 |
|---|---|---|---|---|---|
| A | 5026 | 5674 | 6362 | 7088 | 7854 |

Table for 4 a):

| Marks | 30-40 | 40-50 | 50-60 | 60-70 | 70-80 |
|---|---|---|---|---|---|
| Students | 31 | 42 | 51 | 35 | 31 |

### Module – 3

| | | |
|---|---|---|
| **5 a)** | Solve $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$. | (7m) |
| **b)** | Solve $y'' + 2y' + y = \cosh\left(\frac{x}{2}\right)$. | (6m) |
| **c)** | Solve $y'' - 4y' + 13y = cos2x$. | (7m) |
| | OR | |
| **6 a)** | Solve $(4D^4 - 4D^3 - 23D^2 + 12D + 36)y = 0$. | (7m) |
| **b)** | Solve $(D^4 - 18D^2 + 81)y = 36e^{3x}$. | (6m) |
| **c)** | Solve $(D^2 + 4)y = \sin^2 x$. | (7m) |

## Module – 4

| | | |
|---|---|---|
| **7 a)** | Form a P.D.E. by eliminating the arbitrary constant from the relation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$ | (7m) |
| **b)** | Form a P.D.E. by eliminating the arbitrary function from the relation $z=f(x+y)+g(y+2x)$ | (6m) |
| **c)** | Solve $\frac{\partial^2 z}{\partial x \partial y}=sinxsiny$ for which $\frac{\partial z}{\partial y}=-2siny$ when $x=0$ and $z=0$ when $y$ is an odd multiple of $\frac{\pi}{2}$. | (7m) |
| | OR | |
| **8 a)** | Form a P.D.E. by eliminating the arbitrary function from the relation $z=e^{ax+by}f(ax-by)$. | (7m) |
| **b)** | Form a P.D.E. by eliminating the arbitrary function from the relation $\emptyset(xy+z^2,\ x+y+z)=0.$ | (6m) |
| **c)** | Solve $\frac{\partial^2 z}{\partial y^2}=z$ given that when y=0 $z=e^x\ \ and\ \frac{\partial z}{\partial y}=e^{-x}$. | (7m) |

## Module-5

| | | |
|---|---|---|
| **9 a)** | State and prove addition theorem of probability. | (7m) |
| **b)** | If A and B are independent events, show that the events (i) $\bar{A}$ and $\bar{B}$ (ii) $\bar{A}$ and B (iii) A and $\bar{B}$ are independent. | (6m) |
| **c)** | The probability that 3 students A, B, C solve a problem are 1/2, 1/3, 1/4 respectively. If the problem is simultaneously assigned to all of them, what is the probability that the problem is solved? | (7m) |
| | OR | |
| **10 a)** | State and prove Baye's theorem. | (7m) |
| **b)** | Three machines A, B and C produce respectively 60%, 30%,c10% of the total number of items of a factory. The percentage of defective output of these machines are respectively 2%,3% and 4%. An item is selected at random is found defective. Find the probability that the item was produced by machine C. | (6m) |
| **c)** | A shooter can hit a target in 3 out of 4 shots and another shooter can hit the target in 2 out of 3 shots. Find the probability that the target is being hit (a) when both of then try. (b) By only one shooter. | (7m) |

# Module - 1

1. a) Find the rank of the following matrix by elementary row transformations.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$,
$R_3 \to R_3 - R_1$, $\quad A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

$R_3 \to R_3 + R_2$, $\quad A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The above matrix is in echelon form and has 2 non-zero rows.

$\therefore$ Rank of $A = \rho(A) = 2$.

1. b) Solve the following system of equations by Gauss elimination method

$x+y+z=9$; $x-2y+3z=8$; $2x+y-z=3$.

Solution: The above ~~matrix~~ system can be written in the form of matrix $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & 1 & 3 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$, $[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array}\right]$

$R_3 \to R_2 + (-3)R_3$, $[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 11 & 44 \end{array}\right]$

$\therefore$ Now equations are

$$x+y+z=9 \quad \text{---} (1)$$
$$-3y+2z=-1 \quad \text{---} (2)$$
$$11z=44 \quad \text{---} (3)$$
$$\boxed{z=4}$$

$\therefore$ Equation (2) $\Rightarrow$ $-3y+8=-1$ $\Rightarrow \boxed{y=3}$

Equation (1) $\Rightarrow$ $x+3+4=9$ $\Rightarrow \boxed{x=2}$

$\therefore x=2,\ y=3,\ z=4.$

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1.c) Find the Eigen values and the corresponding Eigen vectors of $A=\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

Solution: The characteristic equation is $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda)-10=0$$

$$\lambda^2-5\lambda-6=0$$

$\therefore \lambda=6$ and $\lambda=-1$ are the Eigen values of A.

The eigen vectors are found from the equation $[A-\lambda I][X]=[0]$.

$$\begin{bmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or
$$\left.\begin{aligned}(1-\lambda)x+2y=0\\ 5x+(4-\lambda)y=0\end{aligned}\right\}-\circledast$$

Case (i): Let $\lambda=6$. $\therefore$ Equation $\circledast$ becomes

$-5x+2y=0$ and $5x-2y=0$

Consider $5x=2y \Rightarrow \frac{x}{2}=\frac{y}{5}=k$ (or) $x=2k,\ y=5k$

If $k=1$ then $x=2,\ y=5$

$\therefore$ Eigen vector when $\lambda=6$ is $X_1=\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

Case (ii): Let $\lambda=-1$. $\therefore$ Equation $\circledast$ is

$2x+2y=0$ and $5x+5y=0$

Consider $2x+2y=0 \Rightarrow \frac{x}{-1}=\frac{y}{1}=k$ (or)

$x=k,\ y=-k$. If $k=1$ then $x=1,\ y=-1$

$\therefore$ Eigen vector when $\lambda=-1$ is $X_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

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2 a) Find the rank of following matrix by reducing into echelon form
$$A=\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Solution: $R_1 \leftrightarrow R_2$,
$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - R_1$ , $A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_2$, $A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The above matrix is in echelon form and has 2 non-zero rows

$\therefore$ Rank of $A = \rho(A) = 2$

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2.b) Solve the following system by Gauss-elimination method, $2x + 5y + 7z = 52$; $2x + y - z = 0$; $x + y + z = 9$

Solution: The matrix form is $AX = B$

$$\begin{bmatrix} 2 & 5 & 7 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 52 \\ 0 \\ 9 \end{bmatrix}$$

The augmented matrix is

$$[A:B] = \begin{bmatrix} 2 & 5 & 7 & : & 52 \\ 2 & 1 & -1 & : & 0 \\ 1 & 1 & 1 & : & 9 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$, $[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 2R_1$, $[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$

$R_3 \to R_3 + 3R_2$ , $[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$

We have $x + y + z = 9$ —— (1)

$-y - 3z = -18$ —— (2)

$-4z = -20 \Rightarrow \boxed{z = 5}$

$\therefore$ Equation (2) $\Rightarrow$ $-y - 15 = -18 \Rightarrow \boxed{y = 3}$

Equation (1) $\Rightarrow$ $x + 3 + 5 = 9 \Rightarrow \boxed{x = 1}$

2. c) Find the Eigen values and Eigen vectors of

$$A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$$

<u>Solution:</u> Characteristic Equation is $|A-\lambda I|=0$

$$\begin{vmatrix}1-\lambda & 1 & 3\\ 1 & 5-\lambda & 1\\ 3 & 1 & 1-\lambda\end{vmatrix}=0$$

$$(1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right]-\left[(1-\lambda)-3\right]+3\left[1-3(5-\lambda)\right]=0$$

$$-\lambda^3+7\lambda^2-36=0 \quad \text{or} \quad \lambda^3-7\lambda^2+36=0$$

$\therefore$ $\lambda=-2,3,6$ are the Eigen values of A.

Eigen vectors are found from $[A-\lambda I][X]=[0]$

$$\begin{bmatrix}1-\lambda & 1 & 3\\ 1 & 5-\lambda & 1\\ 3 & 1 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

or
$$\left.\begin{aligned}(1-\lambda)x+y+3z&=0\\ x+(5-\lambda)y+z&=0\\ 3x+y+(1-\lambda)z&=0\end{aligned}\right\} \quad -(*)$$

<u>Case (i)</u>: Let $\lambda=-2$. Equation $(*)$ is

$3x+y+3z=0$ — (1)

$x+7y+z=0$ — (2)

$3x+y+3z=0$ — (3)

from (1) and (2)
$$\frac{x}{\begin{vmatrix}1 & 3\\ 7 & 1\end{vmatrix}}=\frac{-y}{\begin{vmatrix}3 & 3\\ 1 & 1\end{vmatrix}}=\frac{z}{\begin{vmatrix}3 & 1\\ 1 & 7\end{vmatrix}}$$

$$\frac{x}{-20}=\frac{y}{0}=\frac{z}{20} \quad \text{or} \quad x=1,\ y=0,\ z=-1$$

$\therefore$ Eigen vector when $\lambda=-2$ is $X_1=\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$

<u>Case (ii)</u>: When $\lambda=3$.

Equation $(*)$ is

$-2x+y+3z=0$ — (3)

$x+2y+z=0$ — (4)

$3x+y-2z=0$ — (5)

from ③ & ④, $\dfrac{x}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \dfrac{-y}{\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$

$$\frac{x}{-5} = \frac{-y}{-5} = \frac{z}{-5}$$

(OR) $x=1,\ y=-1,\ z=1$

$\therefore$ Eigen vectors when $\lambda=3$ is $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case (iii): When $\lambda=6$

Equation ⓧ is $-5x+y+3z=0$ — ⑦

$x-y+z=0$ — ⑧

$3x+y-5z=0$ — ⑨

from ⑦ & ⑧ $\dfrac{x}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \dfrac{-y}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$\therefore$ Eigen vectors when $\lambda=6$ is $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

## Module - 2

3 a) The area of a circle (A) corresponding to diameter (D) is given below. Find the area corresponding to diameter 105 using an appropriate interpolation.

| D | 80 | 85 | 90 | 95 | 100 |
|---|---|---|---|---|---|
| A | 5026 | 5674 | 6362 | 7088 | 7854 |

solution: Here 105 is near to 100, Newton's backward interpolation formula is appropriate. D and A correspond to $x$ and $y$.

The backward difference table is

| $x = D$ | $y = A$ | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ |
|---|---|---|---|---|---|
| 80 | 5026 | | | | |
| | | 648 | | | |
| 85 | 5674 | | 40 | | |
| | | 688 | | $-2$ | |
| 90 | 6362 | | 38 | | $4 = \nabla^4 y_n$ |
| | | 726 | | $2 = \nabla^3 y_n$ | |
| 95 | 7088 | | $40 = \nabla^2 y_n$ | | |
| | | $766 = \nabla y_n$ | | | |
| $x_n = 100$ | $y_n = 7854$ | | | | |

By Newton's Backward Interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

$$r = \frac{x - x_n}{h} = \frac{105-100}{5} = 1$$

$$\therefore y(105) = 7854 + 1(766) + \frac{1(2)}{2}(40) + \frac{1(2)(3)}{6}(2) + \frac{(1)(2)(3)(4)}{24}4$$

$$= 7854 + 766 + 40 + 2 + 4$$

$$y(105) = 8666$$

The area A when Diameter $D = 105$ is 8666.

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3.b) Find a real root of $x^3 - 2x - 5 = 0$ correct to three decimal places by using Newton-Raphson method

Solution: Let $f(x) = x^3 - 2x - 5$

Now $f(0) = -5 < 0$, $f(1) = -6 < 0$, $f(2) = -1 < 0$, $f(3) = 16 > 0$

$\therefore$ real root lies in $(2,3)$ and it is near to 2

$\therefore$ Let $x_0 = 2$

Newton Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First iteration is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$= 2 - \frac{f(2)}{f'(2)} = 2 + \frac{1}{10} = 2.1$$

2nd iteration is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$x_2 = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} = 2.0946$$

and $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0946 - \frac{f(2.0946)}{f'(2.0946)}$$

$$= 2.0946 - \frac{(2.0946)^3 - 2(2.0946) - 5}{3(2.0946)^2 - 2}$$

$$x_3 = 2.0946$$

$\therefore$ Required root is $x = 2.0946$.

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3.c) ~~find the fourth root of 12 correct~~

3.c) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using simpsons $1/3^{rd}$ rule taking four equal strips.

Solution: Let $a=0$, $b=1$, $n=4$ $\therefore h = \frac{b-a}{n} = \frac{1}{4}$

$$y = \frac{1}{1+x^2}.$$

Now values of $x$ and $y$ are as follows

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $x_0 = 0$ | $x_1 = x_0 + h$ $= 1/4$ | $x_2 = x_1 + h$ $= 1/2$ | $x_3 = x_2 + h$ $= 3/4$ | $x_4 = 1$ |
| $y = \frac{1}{1+x^2}$ | $y_0 = \frac{1}{1+x_0^2}$ $= 1$ | $y_0 = \frac{1}{1+x_1^2}$ $= 16/17$ | $y_1 = \frac{1}{1+x_2^2}$ $= 4/5$ | $y_3 = \frac{1}{1+x_3^2}$ $= \frac{16}{25}$ | $y_4 = \frac{1}{1+x_4^2}$ $= 1/2$ |

By simpsons $1/3^{rd}$ rule

$$\int_a^b y\,dx = h/3\left((y_0+y_n) + 4(y_1+y_3+y_5+\cdots) + 2(y_2+y_4+\cdots)\right]$$

$$\therefore \int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/4}{3}\left[(y_0+y_4) + 4(y_1+y_3) + 2(y_2)\right]$$

$$= 1/12\left[(1+1/2) + 4\left(\frac{16}{17} + \frac{16}{25}\right) + 2(4/5)\right]$$

$$= 0.7854.$$

4. a) From the following table, find the number of students who have obtained less than 45 marks.

| Marks | 30-40 | 40-50 | 50-60 | 60-70 | 70-80 |
|---|---|---|---|---|---|
| Students | 31 | 42 | 51 | 35 | 31 |

Solution: We shall reconstitute the given table with $f(x)$ representing the number of students less than $x$ marks.

less than 40 marks 31 students
less than 50 marks 73 students
Less than 60 marks 124 students
Less than 70 marks 159 students
Less than 80 marks 190 students

We need to find $f(45)$

| $x$ | $y=f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| $x_0=40$ | $y_0=31$ | $\Delta y_0=42$ | | | |
| 50 | 73 | | $\Delta^2 y_0=9$ | | |
| | | 51 | | $\Delta^3 y_0=-25$ | $\Delta^4 y_0=37$. |
| 60 | 124 | | $-16$ | | |
| | | 35 | | 12 | |
| 70 | 159 | | $-4$ | | |
| | | 31 | | | |
| 80 | 190 | | | | |

By Newton's forward interpolation

$$y = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$$

$$r = \frac{x-x_0}{h} = \frac{45-40}{10} = 0.5$$

$$\therefore y = f(45) = 31 + 0.5(42) + \frac{0.5(0.5-1)\,9}{2} + \frac{0.5(-0.5)(-1.5)}{6}(-25)$$
$$+ \frac{0.5(-0.5)(-1.5)(-2.5)}{24}(37)$$
$$= 48$$

$\therefore$ 48 Students have scored less than 45 marks

4.b) Find the fourth root of 12 correct to three decimal places by Regula-Falsi method

Solution: Let $x = \sqrt[4]{12}$ $\therefore x^4 = 12 \Rightarrow x^4 - 12 = 0$

$\therefore$ Let $f(x) = x^4 - 12$

$f(0) = -12 < 0$, $f(1) = -11 < 0$, $f(2) = 4 > 0$

$\therefore$ root lies in $(1,2)$ and it is near to 2

Now $f(1.9) = 1.0321 > 0$, $f(1.8) = -1.5024 < 0$

$\therefore$ Root lies in $(1.8, 1.9)$

First iteration: $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$$= \frac{1.8f(1.9) + 1.9f(1.8)}{f(1.9) - f(1.8)}$$

$$x_1 = \frac{1.8(1.0321) + 1.9(1.5024)}{1.0321 + 1.5024} = 1.8593$$

Now $f(x_1) = f(1.8593) = (1.8593)^4 - 12 = -0.0492 < 0$

2nd iteration is $x_2 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$$x_2 = \frac{1.8593f(1.9) - 1.9f(1.8593)}{f(1.9) - f(1.8593)}$$

$$= \frac{1.8593(1.0321) - 1.9(-0.0492)}{1.0321 - (-0.0492)} = 1.8612$$

Now $f(x_2) = f(1.8612) = (1.8612)^4 - 12 = -0.00025 < 0$

$\therefore$ Root lies in $(1.8612, 1.9)$

3rd iteration is $x_3 = \dfrac{1.8612f(1.9) - 1.9f(1.8612)}{f(1.9) - f(1.8612)}$

$$= \frac{1.8612(1.0321) - 1.9(-0.00025)}{1.0321 - (-0.00025)}$$

$$x_3 = 1.8612$$

$\therefore$ Required root is $x = 1.8612$

4. c) Evaluate $\int_4^{5.2} \log_e x \, dx$ taking 6 equal strips by applying Weddle's rule.

$\Rightarrow$ $a=4$, $b=5.2$, $n=6$ $\quad \therefore h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$

The values of $x$ and $y$ are as follows

| | | | | |
|---|---|---|---|---|
| $x$ | $x_0 = a$ $= 4$ | $x_1 = x_0 + h$ $= 4.2$ | $x_2 = x_1 + h$ $= 4.4$ | $x_3 = x_2 + h$ $= 4.6$ |
| $y = \log_e x$ | $y_0 = \log x_0$ $= 1.3863$ | $y_1 = \log x_1$ $= 1.4351$ | $y_2 = \log x_2$ $= 1.4816$ | $y_3 = \log x_3$ $= 1.5261$ |

| | | |
|---|---|---|
| $x_4 = x_3 + h$ $= 4.8$ | $x_5 = x_4 + h$ $= 5$ | $x_6 = x_5 + h$ $= 5.2$ |
| $y_4 = \log x_4$ $= 1.5686$ | $y_5 = \log x_5$ $= 1.6094$ | $y_6 = \log x_6$ $= 1.6487$ |

Weddle's rule for $n=6$ is

$$\int_a^b y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$\therefore \int_4^{5.2} \log_e x\,dx = \frac{3(0.2)}{10}\left[1.3863 + 5(1.4351) + 1.4816 + 6(1.5261) + 1.5686 + 5(1.6094) + 1.6487\right]$$

$$\therefore \int_4^{5.2} \log_e x\,dx = 1.8279.$$

5. a) Solve $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$

Solution: We have $(D^3+6D^2+11D+6)y=0$

$\therefore$ A.E. is $m^3+6m^2+11m+6=0$

We shall find the one root by inspection

If $m=-1$ then $(-1)^3+6(-1)^2+11(-1)+6=0$

$\therefore$ $m=-1$ is one root

Now By synthetic division

$$\begin{array}{c|cccc} -1 & 1 & 6 & 11 & 6 \\ & & -1 & -5 & -6 \\ \hline & 1 & 5 & +6 & \boxed{0} \end{array}$$

$\therefore$ Equation is $m^2+5m+6=0$

$(m+2)(m+3)=0$

$m=-2,-3.$

$\therefore$ solution is $y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}$

---

5. b) Solve $y''+2y'+y=\cosh(x/2)$

Solution: We have $(D^2+2D+1)=\cosh(x/2)$

A.E. is $m^2+2m+1=0 \Rightarrow m=-1,-1$

$\therefore$ $y_c=(c_1+c_2x)e^{-x}.$

Now $y_p=\frac{\cosh(x/2)}{D^2+2D+1}=\frac{1}{2}\left[\frac{e^{x/2}}{D^2+2D+1}+\frac{e^{-x/2}}{D^2+2D+1}\right]$

Type(1), $a=1/2$ , Type(1), $a=-1/2$

Replace $D$ by $1/2$ , Replace $D$ by $-1/2$

$=\frac{1}{2}\left[\frac{e^{x/2}}{1/4+1+1}+\frac{e^{-x/2}}{1/4-1+1}\right]$

$y_p=\frac{2}{9}e^{x/2}+2e^{-x/2}$

$\therefore$ Complete solution is $y=y_c+y_p$

$y=(c_1+c_2x)e^{-x}+\frac{2}{9}e^{x/2}+2e^{-x/2}$

5. c) Solve $y'' - 4y' + 13y = \cos 2x$

Solution: We have $(D^2 - 4D + 13)y = \cos 2x$

A.E. is $m^2 - 4m + 13 = 0 \Rightarrow m = 2 \pm 3i$

$\therefore y_c = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$

Now $y_p = \dfrac{\cos 2x}{D^2 - 4D + 13}$ Type ② $a = 2$ Replace $D^2$ by $-a^2 = -4$

$= \dfrac{\cos 2x}{-4 - 4D + 13} = \dfrac{\cos 2x}{9 - 4D} \times \left(\dfrac{9+4D}{9+4D}\right)$

$= \dfrac{(9+4D)\cos 2x}{81 - 16D^2}$, Replace $D^2$ by $-a^2 = -4$

$= \dfrac{9\cos 2x - 8\sin 2x}{81 - 16(-4)} = \dfrac{9\cos 2x - 8\sin 2x}{145}$

$\therefore$ Complete solution is $y = y_c + y_p$

$y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) + \dfrac{9\cos 2x - 8\sin 2x}{145}$.

---

6 a). Solve $(4D^4 - 4D^3 - 23D^2 + 12D + 36)y = 0$

Solution: A.E. is $4m^4 - 4m^3 - 23m^2 + 12m + 36 = 0$

If $m = 2$ then $4(2)^4 - 4(2)^3 - 2(2)^2 + 12(2) + 36 = 0$

$\therefore$ $m = 2$ is a root by inspection.

Now by synthetic division

$$\begin{array}{c|ccccc} 2 & 4 & -4 & -23 & 12 & 36 \\ & & 8 & 8 & -30 & -36 \\ \hline & 4 & 4 & -15 & -18 & \boxed{0} \end{array}$$

Now $4m^3 + 4m^2 - 15m - 18 = 0$

Again if $m = 2 \Rightarrow 4(8) + 4(4) - 15(2) - 18 = 0$

$m = 2$ is a root by inspection.

Again by synthetic division

$$\begin{array}{r|rrrr} 2 & 4 & 4 & -15 & -18 \\ & & 8 & 24 & 18 \\ \hline & 4 & 12 & 9 & \boxed{0} \end{array}$$

$\Rightarrow 4m^2+12m+9=0$

$(2m+3)^2=0$

$\Rightarrow m=-3/2, -3/2$

$\therefore y=(C_1+C_2x)e^{2x}+(C_3+C_4x)e^{-3/2x}$ is the g.s.

---

6.b) Solve $(D^4-18D^2+81)y=36e^{3x}$

Solution: A.E. is $m^4-18m^2+81=0$

$(m^2-9)^2=0$ (or) $((m-3)(m+3))^2=0$

$(m-3)^2=0, (m+3)^2=0$

$m=3,3,-3,-3$

$\therefore y_c=(C_1+C_2x)e^{3x}+(C_3+C_4x)e^{-3x}$

Now $y_p=\dfrac{36e^{3x}}{D^4-18D^2+81}$ Type (1) $a=3$ Replace $D$ by 3

$=\dfrac{36e^{3x}}{3^4-18(9)+81}$ But Denominator is 0

$y_p=\dfrac{x\,36e^{3x}}{4D^3-36D}=\dfrac{x\,36e^{3x}}{4(27)-36(3)}$, Again $Dr=0$

$\therefore y_p=\dfrac{36x^2e^{3x}}{12D^2-36}=\dfrac{36x^2e^{3x}}{12(9)-36}=\dfrac{x^2e^{3x}}{2}$

$\therefore$ Complete solution is

$y=y_c+y_p$

$y=(C_1+C_2x)e^{3x}+(C_3+C_4x)e^{-3x}+\dfrac{x^2e^{3x}}{2}$.

6. c) Solve $(D^2+4)y = \sin^2 x$

Solution: A.E. is $m^2+4=0 \Rightarrow m = \pm 2i$

$\therefore y_c = c_1\cos 2x + c_2 \sin 2x$

Also $y_p = \dfrac{\sin^2 x}{D^2+4} = \dfrac{1}{2}\left(\dfrac{1-\cos 2x}{D^2+4}\right)$

$= \dfrac{1}{2}\left[\dfrac{1}{D^2+4} + \left(\dfrac{-\cos 2x}{D^2+4}\right)\right] = \dfrac{1}{2}(P_1+P_2)$

Now $P_1 = \dfrac{1}{D^2+4}$ Type ① $a=0$. Replace $D$ by $0$

$P_1 = \dfrac{1}{0+4} = \dfrac{1}{4}$.

and $P_2 = \dfrac{-\cos 2x}{D^2+4}$, Type ② $a=2$ Replace $D^2$ by $-4$

But denominator is $0$

$\therefore P_2 = -x\dfrac{\cos 2x}{2D} = -\dfrac{x}{2}\displaystyle\int \dfrac{\cos 2x}{D}\,dx$

$P_2 = \dfrac{-x\sin 2x}{4}$.

$\therefore y_p = \dfrac{1}{2}(P_1+P_2) = \dfrac{1}{2}\left[\dfrac{1}{4} - \dfrac{x\sin 2x}{4}\right]$

$y_p = \dfrac{1}{8} - \dfrac{x\sin 2x}{8}$

$\therefore$ Complete solution is $y = y_c + y_p$

$y = c_1\cos 2x + c_2\sin 2x + \dfrac{1}{8} - \dfrac{x\sin 2x}{8}$.

# Module-4

7.a) Form a P.D.E. by eliminating the arbitrary constant from the relation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solution:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{——— (1)}$$

Differentiate (1) w.r.t $x$ and $y$ partially

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \Rightarrow \frac{x}{a^2}+\frac{z}{c^2}\frac{\partial z}{\partial x}=0 \quad \text{— (2)}$$

$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \Rightarrow \frac{y}{b^2}+\frac{z}{c^2}\frac{\partial z}{\partial y}=0 \quad \text{— (3)}$$

Since there are 3 arbitrary constants, we differentiate further.

diff (2) partially w.r.t. $x$

$$\frac{1}{a^2}+\frac{1}{c^2}\left(\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}\right)=0 \quad \text{——— (4)}$$

From (2), $\frac{1}{a^2}=-\frac{z}{c^2x}\frac{\partial z}{\partial x}$

$\therefore$ Equation (4) is $-\frac{z}{c^2x}\frac{\partial z}{\partial x}+\frac{1}{c^2}\left(\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}\right)=0$

$$\frac{z}{x}\frac{\partial z}{\partial x}=\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}$$

$$z\frac{\partial z}{\partial x}=x\left(\frac{\partial z}{\partial x}\right)^2+xz\frac{\partial^2 z}{\partial x^2}$$ is the

required p.d.e

7.b) Form a P.D.E. by eliminating arbitrary function from $z = f(y+x) + g(y+2x)$

Solution: Now $\frac{\partial z}{\partial x} = f'(x+y) + 2g'(y+2x)$

$\frac{\partial z}{\partial y} = f'(x+y) + g'(y+2x)$

$\frac{\partial^2 z}{\partial x^2} = f''(x+y) + 4g''(y+2x)$ —— ①

$\frac{\partial^2 z}{\partial x \partial y} = f''(x+y) + 2g''(y+2x)$ —— ②

$\frac{\partial^2 z}{\partial y^2} = f''(x+y) + g''(y+2x)$ —— ③

Now ① - ② gives $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = 2g''(y+2x)$ —— ④

and ② - ③ gives $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = g''(y+2x)$ —— ⑤

Now dividing ④ by ⑤ we get

$$\frac{\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y}}{\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2}} = 2$$

(OR) $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$ is the required P.D.E.

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7.c) Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$ and $z=0$ if $y$ is an odd multiple of $\pi/2$

Solution: $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ (or) $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \cdot \sin y$

Integrating w.r.t $x$ treating $y$ as constant

$\frac{\partial z}{\partial y} = \sin y \int \sin x \, dx + f(y) = -\sin y \sin x + f(y)$ —— ①

Integrating w.r.t $y$ treating $x$ as constant

$z = -\cos x \int \sin y\, dy + \int f(y)\, dy + g(x)$

$z = \cos x \cos y + \int f(y)\, dy + g(x)$ —— (2)

Also by data $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$

$\therefore$ Equation (1) $\Rightarrow$ $-2\sin y = -\sin y + f(y) \Rightarrow f(y) = -\sin y$

Also $\int f(y)\, dy = \int -\sin y\, dy = \cos y$.

$\therefore$ Equation (2) is $z = \cos x \cos y + \cos y + g(x)$ —— (3)

2nd condition is $z = 0$ if $y = (2n+1)\pi/2$

$\therefore$ Equation (2) is $0 = \cos x \cos(2n+1)\pi/2 + \cos(2n+1)\frac{\pi}{2} + g(x)$

$0 = 0 + 0 + g(x) \Rightarrow g(x) = 0$

$\therefore$ Solution of the P.D.E. is

$z = \cos x \cos y + \cos y = \cos y(\cos x + 1)$.

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8 a) Form a P.D.E. by eliminating arbitrary function from $z = e^{ax+by} f(ax-by)$.

Solution: $z = e^{ax+by} f(ax-by)$

Now $\frac{\partial z}{\partial x} = a e^{ax+by} f'(ax-by) + a e^{ax+by} f(ax-by)$

(or) $\frac{\partial z}{\partial x} = a e^{ax+by} f'(ax-by) + az$ —— (2)

Also $\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + b e^{ax+by} f(ax-by)$

$= -b e^{ax+by} f'(ax-by) + bz$ —— (3)

Multiplying (2) by 'b', (3) by 'a' and adding

$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$ is the required P.D.E.

8.b) Form a P.D.E by eliminating arbitrary function from $\phi(xy+z^2, x+y+z)=0$

Solution We have by data $\phi(u,v)=0$ —— ①

where $u=xy+z^2$ and $v=x+y+z$

$$\frac{\partial u}{\partial x}=y+2z\frac{\partial z}{\partial x} \quad \text{and} \quad \partial v/\partial x=1+\partial z/\partial x$$

$$\frac{\partial u}{\partial y}=x+2z\frac{\partial z}{\partial y} \quad \quad \partial v/\partial y=1+\partial z/\partial y$$

Let us differentiate ① w.r.t $x$ and $y$ by applying chain rule

$$\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x}=0 \quad \text{(or)} \quad \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}=-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x} \quad —②$$

$$\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y}=0 \quad \text{(or)} \quad \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}=-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y} \quad —③$$

Dividing ② by ③, we get

$$\frac{\partial u/\partial x}{\partial u/\partial y}=\frac{\partial v/\partial x}{\partial v/\partial y}$$

$$\frac{y+2z\,\partial z/\partial x}{x+2z\,\partial z/\partial y}=\frac{1+\partial z/\partial x}{1+\partial z/\partial y}$$

$$y+2z\,\partial z/\partial x+y\frac{\partial z}{\partial y}+2z\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}=x+2z\frac{\partial z}{\partial y}+x\frac{\partial z}{\partial x}+2z\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$\therefore \frac{\partial z}{\partial x}(x-2z)\mp\frac{\partial z}{\partial y}(y-2z)+x-y=0$$

is the required P.D.E

8. c) Solve $\frac{\partial^2 z}{\partial y^2} = z$ given that when $y=0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$.

<u>Solution</u> Assume the given p.d.e in the form of O.D.E.

$$\frac{d^2 z}{dy^2} = z \quad (\text{or}) \quad \frac{d^2 z}{dy^2} - z = 0 \quad (\text{or}) \quad (D^2-1)z = 0$$

A.E. is $m^2 - 1 = 0 \Rightarrow m = 1, -1$

$\therefore$ Solution of p.d.e is

$$z = f(x)e^{y} + g(x)e^{-y} \quad \text{—— (1)}$$

Also $$\frac{\partial z}{\partial y} = f(x)e^{y} - g(x)e^{-y} \quad \text{—— (2)}$$

Given that $z = e^x$ when $y=0$ and $\frac{\partial z}{\partial y} = e^{-x}$ when $y=0$.

$\therefore$ Equations (1) & (2) will become

$$e^x = f(x) + g(x) \quad \text{—— (3)}$$

$$e^{-x} = f(x) - g(x) \quad \text{—— (4)}$$

Now (3) + (4) gives $2f(x) = e^x + e^{-x}$

$$\therefore f(x) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and (3) − (4) gives $$g(x) = \frac{e^x - e^{-x}}{2}$$

$$g(x) = \sinh x.$$

$\therefore$ solution is $z = (\cosh x)e^{y} + (\sinh x)e^{-y}$.

## Module - 5

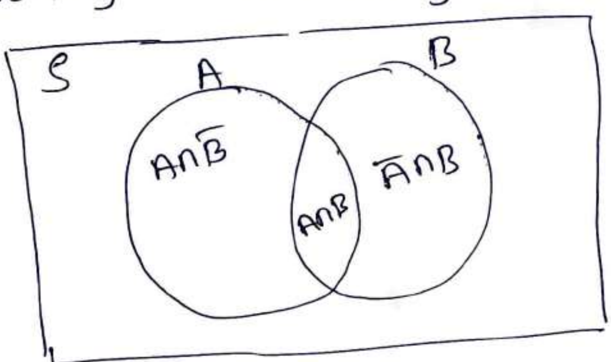
9 a) State and prove addition theorem of probability

Solution: statement:

If A and B are any two events of S which are not mutually exclusive then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: We prove the result using the following Venn diagram.




From the figure, $A = (A \cap \bar{B}) \cup (A \cap B)$

$B = (\bar{A} \cap B) \cup (A \cap B)$

$\Rightarrow P(A) = P(A \cap \bar{B}) + P(A \cap B)$, Since $A \cap \bar{B}$ and $A \cap B$ are disjoint

$P(B) = P(\bar{A} \cap B) + P(A \cap B)$, since $\bar{A} \cap B$ and $A \cap B$ are disjoint.

$\therefore$ from the axioms of probability

$$P(A) + P(B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B) + P(A \cap B)$$

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

Thus $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

9 b). If $A$ and $B$ are independent events, show that the events
(i) $\bar{A}$ and $\bar{B}$ (ii) $\bar{A}$ and $B$ (iii) $A$ and $\bar{B}$ are independent.

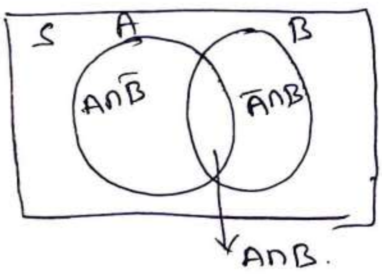
**Solution:** (i) We have by De-morgan's law

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Now
$$\begin{aligned}
P(\bar{A} \cap \bar{B}) &= P(\overline{A \cup B}) \\
&= 1 - P(A \cup B) \\
&= 1 - P(A) - P(B) + P(A \cap B) \\
&= 1 - P(A) - P(B) + P(A) \cdot P(B) \\
&= [1 - P(A)][1 - P(B)] \\
&= P(\bar{A}) \cdot P(\bar{B})
\end{aligned}$$

Thus $\bar{A}$ and $\bar{B}$ are independent

(ii) From the figure,




$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$\begin{aligned}
P(\bar{A} \cap B) &= P(B) - P(A \cap B) = P(B) - P(A) \cdot P(B) \\
&= P(B)[1 - P(A)] = P(B) \cdot P(\bar{A})
\end{aligned}$$

$\therefore$ $\bar{A}$ and $B$ are independent

(iii) Also from the figure

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

$$P(A) = P(A \cap \bar{B}) + P(A \cap B)$$

$$\begin{aligned}
P(A \cap \bar{B}) &= P(B) - P(A \cap B) = P(B) - P(A) \cdot P(B) \\
&= P(B)(1 - P(A)) \\
&= P(B)\, P(\bar{A})
\end{aligned}$$

$\therefore$ $A$ and $\bar{B}$ are independent.

9.c) The probability that 3 students A, B, C solve a problem are 1/2, 1/3, 1/4 respectively If the problem is simultaneously assigned to all of them, what is the probability that the problem is solved?

Solution: probability that A can solve the problem $P(A) = 1/2$

probability that B can solve the problem $P(B) = 1/3$

Probability that C can solve the problem $P(C) = 1/4$

P(problem solved) = 1 − P(problem being unsolved by A, B, C)

$$P(E) = 1 - P(\bar{E})$$

$P(\bar{E})$ is the probability that the problem is not solved

$$\therefore P(\bar{E}) = P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$$
$$= (1-P(A))(1-P(B))(1-P(C))$$
$$= \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \tfrac{3}{4} = \tfrac{1}{4}$$

Hence $P(E) = 1 - 1/4 = 3/4$.

probability that the problem is solved is 3/4

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10.a) State and prove Baye's theorem

Solution: Statement: Let $A_1, A_2, \ldots A_n$ be a set of exhaustive, mutually exclusive events of a sample space S with $P(A_i) \neq 0$ for each $i$, If A is any other event associated with $A_i$, $(A \subset \bigcup_{i=1}^{n} A_i)$

with $P(A) \neq 0$ then

$$P(A_i/A) = \frac{P(A_i)\,P(A/A_i)}{\sum_{i=1}^{n} P(A_i)\,P(A/A_i)}$$

proof: We have, $S = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$ and $A \subset S$

$\therefore A = S \cap A = (A_1 \cup A_2 \cup \cdots A_n) \cap A$

Using distributive law in the RHS

$A = (A_1 \cap A) \cup (A_2 \cap A) \cup \cdots \cup (A_n \cap A)$

Since $A_i \cap A$ for $i=1$ to $n$ are mutually exclusive, we have by applying the addition rule of probability,

$P(A) = P(A_1 \cap A) + P(A_2 \cap A) + \cdots + P(A_n \cap A)$

Now applying multiplication rule onto each term in the RHS,

$P(A) = P(A_1) \cdot P(A/A_1) + P(A_2) \cdot P(A/A_2) + \cdots + P(A_n) \cdot P(A/A_n)$

$$P(A) = \sum_{i=1}^{n} P(A_i)\,P(A/A_i) \quad \text{——} \quad (1)$$

The conditional probability of $A_i$ for any $i$ given $A$, is defined by

$$P(A_i/A) = \frac{P(A_i \cap A)}{P(A)} = \frac{P(A_i) \cdot P(A/A_i)}{P(A)}$$

Using (1),

$$P(A_i/A) = \frac{P(A_i) \cdot P(A/A_i)}{\sum_{i=1}^{n} P(A_i)\,P(A/A_i)}$$

10.4) Three machines A, B and C produce respectively 60%, 30%, 10% of the total number of items of a factory. The percentages of defective output of these machines are respectively 2%, 3% and 4%. An item is selected at random and is found defective. Find the probability that the item was produced by Machine C.

Solution: Let A, B, C stand for the events of selection of an item from machines A, B, C.

$\therefore P(A) = 60\% = 0.6$, $P(B) = 30\% = 0.3$, $P(C) = 10\% = 0.1$

Let D be the event of selection of a defective item then

$P(D/A) = 2\% = 0.02$, $P(D/B) = 3\% = 0.03$

$P(D/C) = 4\% = 0.04$.

To find the probability that a selected item is produced from the machine C, we need to find $P(C/D)$.

By Baye's theorem

$$P(C/D) = \frac{P(C) \cdot P(D/C)}{P(A) \cdot P(D/A) + P(B) \cdot P(D/B) + P(C) \cdot P(D/C)}$$

$$= \frac{(0.1)(0.04)}{(0.6)(0.02) + (0.3)(0.03) + (0.1)(0.04)}$$

$$P(C/D) = 0.16$$

10. c) A shooter can hit a target in 3 out of 4 shots and another shooter can hit the target in 2 out of 3 shots. Find the probability that the target is being hit

ⓐ when both of them try ⓑ by only one shooter

Solution: Let $S_1$ and $S_2$ be the events that the shooters 1 and 2 hit the target.

$P(S_1) = 3/4, \quad P(S_2) = 2/3$

a) When both of them try

$$P(S_1 \cup S_2) = P(S_1) + P(S_2) - P(S_1 \cap S_2)$$

But $S_1$ and $S_2$ are independent

$$\therefore P(S_1 \cup S_2) = P(S_1) + P(S_2) - P(S_1) \cdot P(S_2)$$

$$= \frac{3}{4} + 2/3 - \frac{3}{4} \cdot 2/3 = 11/12$$

$$P(S_1 \cup S_2) = 11/12$$

b) By only one shooter

$(S_1 \cap \bar{S_2}) \cup (\bar{S_1} \cap S_2)$

$$\therefore P[(S_1 \cap \bar{S_2}) \cup (\bar{S_1} \cap S_2)] = P(S_1 \cap \bar{S_2}) + P(\bar{S_1} \cap S_2)$$

$$= P(S_1) \cdot P(\bar{S_2}) + P(\bar{S_1}) \cdot P(S_2)$$

$$= \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3}$$

$$= 5/12$$

$\therefore$ Required probability is $5/12$.