SPH

18MAT21

Second Semester B.E. Degree Examination, June/July 2019
Advanced Calculus and Numerical Methods

Time: 3 hrs. Max. Marks: 100

Note: Answer any FIVE full questions, choosing ONE full question from each module.

Module-1

1 a. If $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$, find div $\vec{F}$ and curl $\vec{F}$. **(06 Marks)**

b. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$. **(07 Marks)**

c. Find the value of a, b, c such that $\vec{F} = (axy + bz^3)\hat{i} + (3x^2 - cz)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational, also find the scalar potential ϕ such that $\vec{F} = \nabla\phi$. **(07 Marks)**

OR

2 a. Find the total work done in moving a particle in the force field $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$. **(06 Marks)**

b. Using Green's theorem, evaluate
$\int_C (xy + y^2)dx + x^2dy$, where C is bounded by $y = x$ and $y = x^2$. **(07 Marks)**

c. Using Divergence theorem, evaluate $\int_S \vec{F} ds$, where $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. **(07 Marks)**

Module-2

3 a. Solve $(D^2 - 3D + 2)y = 2x^2 + \sin 2x$. **(06 Marks)**

b. Solve $(D^2 + 1)y = \sec x$ by the method of variation of parameter. **(07 Marks)**

c. Solve $x^2y'' - 4xy' + 6y = \cos(2 \log x)$ **(07 Marks)**

OR

4 a. Solve $(D^2 - 4D + 4)y = e^{2x} + \sin x$. **(06 Marks)**

b. Solve $(x+1)^2y'' + (x+1)y' + y = 2\sin[\log_e(x+1)]$ **(07 Marks)**

c. The current i and the charge q in a series containing an inductance L, capacitance C, emf E, satisfy the differential equation $L\frac{d^2q}{dt^2} + \frac{q}{C} = E$. Express q and i interms of 't' given that L, C, E are constants and the value of i and q are both zero initially. **(07 Marks)**

Module-3

5 a. Form the partial differential equation by elimination of arbitrary function from $\phi(x + y + z,\ x^2 + y^2 + z^2) = 0$ **(06 Marks)**

b. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$ **(07 Marks)**

c. Derive one dimensional heat equation in the standard form as $\frac{\partial U}{\partial t} = C^2 \frac{\partial^2 U}{\partial x^2}$. **(07 Marks)**

OR

6 a. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ such that $z = e^y$ where $x = 0$ and $\frac{\partial z}{\partial x} = 1$ when $x = 0$. **(06 Marks)**

b. Solve $(mz - ny)\frac{\partial z}{\partial x} + (nx - \ell z)\frac{\partial z}{\partial y} = \ell y - mx$ **(07 Marks)**

c. Find all possible solutions of one dimensional wave equation $\frac{\partial^2 U}{\partial t^2} = C^2 \frac{\partial^2 U}{\partial x^2}$ using the method of separation of variables. **(07 Marks)**

Module-4

7 a. Discuss the nature of the series $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} x^n$. **(06 Marks)**

b. With usual notation prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ **(07 Marks)**

c. If $x^3 + 2x^2 - x + 1 = aP_3 + bP_2\ cP_1 + dP_0$, find a, b, c and d using Legendre's polynomial. **(07 Marks)**

OR

8 a. Discuss the nature of the series

$$\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{3.4} + \ldots\ldots\ldots$$

Correction ,

(06 Marks)

b. Obtain the series solution of Legendre's differential equation in terms of $P_n(x)$

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

(07 Marks)

c. Express $x^4 - 3x^2 + x$ interms of Legendre's polynomial. **(07 Marks)**

Module-5

9 a. Find the real root of the equation $x\sin x + \cos x = 0$ near $x = \pi$ using Newton-Raphson method. Carry out 3 iterations. **(06 Marks)**

b. From the following data, find the number of students who have obtained (i) less than 45 marks (ii) between 40 and 45 marks.

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 - 80
No. of Students	31	42	51	35	31

(07 Marks)

c. Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ using Simpson's $\frac{3}{8}^{th}$ rule by taking 7 ordinates. **(07 Marks)**

OR

10 a. Find the real root of the equation $x \log_{10} x = 1.2$ which lies between 2 and 3 using Regula-Falsi method. **(06 Marks)**

b. Using Lagrange's interpolation formula, find y at x = 4, for the given data:

x	0	1	2	5
y	2	3	12	147

(07 Marks)

c. Evaluate $\int_4^{5.2} \log_e x\, dx$ using Weddle's rule by taking six equal parts. **(07 Marks)**

II Semester B. E. Degree Examination June/July 2019

Advanced Calculus & Numerical Methods

Detailed Solution

14/07/2021.

1 a) Given $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$

$$= \frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)\,i + \frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)\,j + \frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$= (3x^2-3yj)\,i + (3y^2-3xz)\,j + (3z^2-3xy)\,k$$

$$\therefore \text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right] - j\left[\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right]$$

$$+k\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$= i[3x-3x] - j[-3y+3y] + k[3z-3z]$$

$$\therefore \text{curl}\,\vec{F} = 0$$

P.T.O

b) Let ϕ_1 and ϕ_2 be the two given surfaces

$$\phi_1 = x^2+y^2+z^2-9 \text{ and } \phi_2 = z-x^2+y^2+3$$

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x} i + \frac{\partial\phi_1}{\partial y} \cdot j + \frac{\partial\phi_1}{\partial z} k$$

$$= 2xi+2yj+2zk$$

$$(\nabla\phi_1)_{(2,-1,2)} = 4i-2j+4k$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x} i + \frac{\partial\phi_2}{\partial y} j + \frac{\partial\phi_2}{\partial z} k$$

$$= -2xi-2yj+k.$$

$$(\nabla\phi_2)_{(2,-1,2)} = -4i+2j+k$$

Angle between the surfaces is

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

$$= \left| \frac{(4)(-4)+(-2)(2)+(4)(1)}{\sqrt{(4)^2+(-2)^2(4)^2} \cdot \sqrt{(-4)^2+(2)^2+(1)^2}} \right|$$

$$= \left| \frac{-16}{\sqrt{36}\sqrt{21}} \right|$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

c) Given

$\vec{F} = (axy + bz^3)i + (3x^2 - cz)j + (3xz^2 - y)k$ is

irrotational

ie $\nabla \times \vec{F} = 0$

$$\Rightarrow \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy + bz^3 & 3x^2 - cz & 3xz^2 - y \end{vmatrix} = 0$$

$$= i\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - cz)\right] - j\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(axy + bz^3)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(3x^2 - cz) - \frac{\partial}{\partial y}(axy + bz^3)\right]$$

$$\Rightarrow i[(-1) + c] - j[3z^2 - 3bz^2] + k[6x - ax] = 0$$

$$\Rightarrow c - 1 = 0, \quad 3z^2 - 3bz^2 = 0, \quad 6x - ax = 0$$

$$\Rightarrow a = 6, \; b = 1, \; c = 1$$

Consider $\vec{F} = \nabla \phi$

$$\Rightarrow (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$\frac{\partial \phi}{\partial x} = 6xy + z^3, \Rightarrow \phi = 3x^2 y + xz^3 + \phi_1(y, z)$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z \Rightarrow \phi = 3x^2 - yz + \phi_2(x, z)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y \Rightarrow \phi = xz^3 - yz + \phi_3(x, y)$$

To get the unique expression for ϕ.

Let $\phi_1(yz) = -yz \quad \phi_2(x, z) = xz^3, \quad \phi_3(x, y) = 3x^2 y$

$$\therefore \phi = 3x^2 y - yz + xz^3$$

P.T.O.

2

a) Given $\vec{F} = 3xyi - 5zj + 10xk$

and $\vec{r} = xi + yj + zk$

$d\vec{r} = dx\,i + dy\,j + dz\,k$

$\vec{F}\cdot d\vec{r} = (3xyi - 5zj + 10xk)\cdot(dx\,i + dy\,j + dz\,k)$

$\vec{F}\cdot d\vec{r} = 3xy\,dx - 5z\,dy + 10x\,dz$

Since $x = t^2+1$, $y = 2t^2$ and $z = t^3$

$dx = 2t\,dt$, $dy = 4t\,dt$, $dz = 3t^2dt$

$\therefore \vec{F}\cdot d\vec{r} = 3(t^2+1)(2t^2)(2t\,dt) - 5(t^3)(4t)\,dt + 10(t^2+1)(3t^2dt)$

$\therefore \vec{F}\cdot d\vec{r} = 12(t^5+t^3)\,dt - 20t^4\,dt + 30(t^4+t^2)\,dt$

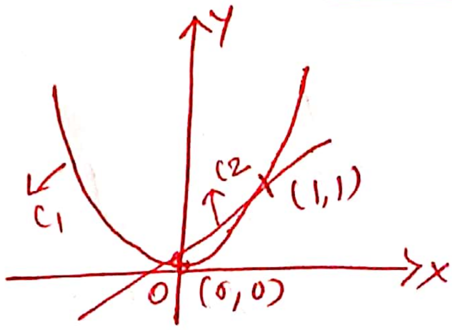
$\therefore \int_{t=1}^{2} \vec{F}\cdot d\vec{r} = \int_{t=1}^{2} (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$

$= \left[12\frac{t^6}{6} + 10\frac{t^5}{5} + 12\cdot\frac{t^4}{4} + 30\frac{t^3}{3}\right]_{t=1}^{2}$

$= \left[2(2^6 - 1^6) + 2(2^5 - 1^5) + 3((2)^4 - 1^4) + 10(2^3 - 1^3)\right]$

$\therefore \int_{t=1}^{2} \vec{F}\cdot d\vec{r} = 933.$

b) Here $M = xy + y^2$, $N = x^2$



$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = x - 2y$

In the given region x varies from $x = 0$, to $x = 1$ and 'y' varies from $y = x^2$ to x

$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} (x - 2y)\,dx\,dy$

$$= \int_{x=0}^{1} (xy - y^2)_{x^2}^{x}\, dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}\right) dx\,dy = \frac{-1}{20} \longrightarrow (1)$$

Consider $\oint [M dx + N dy] = \oint_{C_1} (M dx + N dy) + \oint_{C_2} (M dx + N dy)$

Along C_1 : $y = x^2 \Rightarrow dy = 2x\,dx$

Along C_2 ; $y = x \Rightarrow dx = dy$

$$\oint_{C_1} (M dx + N dy) = \int_0^1 [3x^3 + x^4]\,dx = \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

$$\oint_{C_2} (M dx + N dy) = \int_1^0 (3x^2 dx) = [x^3]_1^0 = -1$$

$$\therefore \oint_C (M dx + N dy) = \frac{19}{20} - 1 = \frac{-1}{20} \longrightarrow (2)$$

From Eq^n (1) & (2)

$$\oint (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}\right) dx\,dy$$

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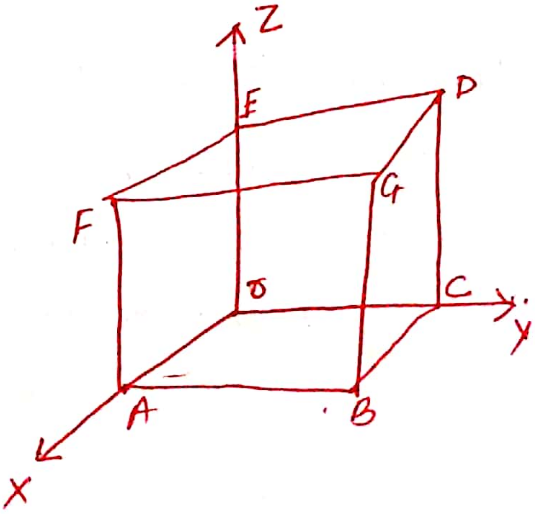
c) Given $\vec{F} = (x^2 - yz)i + (y^2 - xz)j + (z^2 - xy)k$.

$$\operatorname{div} \vec{F} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 2(x+y+z)$$

$$\iiint_E \operatorname{div} F\, dv = \int_0^c \int_0^b \int_0^a 2(x+y+z)\,dx\,dy\,dz$$

$$= \frac{2ab}{2}\left[az + bz + z^2\right]_0^c$$

$$= (abc)(a+b+c) \longrightarrow (1)$$



$$\iint F\cdot n\, ds = \sum_{i=1}^{4} \oint_{S_i} F\cdot n\, ds \longrightarrow (2)$$

On the surface $S_1 = OABC$ $(z=0,\ n=-k)$

$$\oint_{S_1} F\cdot n\, ds = -\int_{S_1} (z^2 - xy)\, dx\, dy = -\int_0^b \int_0^a (-xy\, dx\, dy) = \frac{a^2b^2}{4}$$

On the surface $S_2 = OCDE$ $(x=0,\ n=-i)$

$$\oint_{S_2} F\cdot n\, ds = -\int_{S_2} (x^2 - yz) dy\, dz = \int_0^c \int_0^b (-yz)\, dy\, dz = \frac{b^2c^2}{4}$$

On the surface $S_3 = OAFE$ $(y=0,\ n=-j)$

$$\oint_{S_3} F\cdot n\, ds = -\int_{S_3} (y^2 - zx) dx\, dz = -\int_0^c \int_0^a -(zx)\, dx\, dz = \frac{a^2c^2}{4}$$

on the surface $S_4 = ABGF$ $(x=a,\ n=i)$ we have

$$\oint_{S_4} F\cdot n\, ds = -\int_{S_4} (x^2 - yz)\, dy\, dz = \int_0^c \int_0^b (a^2 - yz)\, dy\, dz = a^2bc - \frac{b^2c^2}{4}$$

Similarly on the surface $S_5 = BCDG$ $(y=b,\ n=j)$

$$\oint_{S_5} F\cdot n\, ds = ab^2c - \frac{a^2c^2}{4} \text{ and } S_6 = (z=c,\ n=k)$$

$$\oint_{S_6} F\cdot n\, ds = abc^2 - \frac{a^2b^2}{4}$$ substituting these in Eqn (2) we get

$$\oint_{C} F\cdot n\, ds = abc(a+b+c)$$

3.

a) Given $(D^2 - 3D + 2)y = 2x^2 + \sin 2x$

A.E is $m^2 - 3m + 2 = 0$

$m^2 - m - 2m + 2 = 0$

$m(m-1) - 2(m-1) = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

The roots are real & distinct.

$C.F = c_1 e^{x} + c_2 e^{2x} \longrightarrow (1)$

$P.I = \frac{1}{f(D)} \cdot X = \frac{1}{f(D)} (2x^2) + \frac{1}{f(D)} \sin 2x = P_1 + P_2$

Consider

$P_1 = \frac{1}{f(D)} \cdot 2x^2$

By the method of division

$$\begin{array}{r|l} & x^2+3x+7/2 \\ 2-3D+D^2 & 2x^2 \\ & 2x^2-6x+2 \\ \hline & 6x-2 \\ & 6x-9 \\ \hline & 7 \\ & 7 \\ \hline & 0 \end{array}$$

$\therefore P_1 = x^2+3x+7/2$

$P_2 = \frac{1}{D^2-3D+2} \sin 2x$ Replace D^2 by $-a^2 = -4$

$= \frac{1}{-4-3D+2} \cdot \sin 2x = \frac{-1}{(3D+2)} \cdot \frac{(3D-2)}{(3D-2)} \cdot \sin 2x$

$P_2 = \frac{(2-3D)\sin 2x}{(9D^2-4)_{D^2 \to -4}} = \frac{-1}{40}(2\sin 2x - 6\cos 2x)$

$\therefore P.I = x^2+3x+7/2 - \frac{1}{40}[2\sin 2x - 6\cos 2x]$

$\therefore G.S = C.F + P.I.$

b) Given $(D^2+1)y = \sec x$

A.E is $m^2+1=0$

$m^2 = -1 \Rightarrow m = \pm i$ The roots are imaginary.

$\therefore CF = c_1 \cos x + c_2 \sin x \longrightarrow (1)$

Let $y = v_1 \cos x + u_2 \sin x$ be the P.I.

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \Rightarrow \boxed{W=1}$

8

$$V_1 = -\int \frac{y_2 \cdot X}{W}\,dx = -\int \frac{\sec x \sin x\,dx}{1} = -\log \sec x$$

$$V_2 = \int \frac{y_1 X}{W}\,dx = \int \frac{\cos x \cdot \sec x\,dx}{1} = x$$

$\therefore$ $y = C.F + P.I.$

$$y = C_1 \cos x + C_2 \sin x - \log \sec x + x$$

c) Given $x^2 y'' - 4xy' + 6y = \cos(2\log x)$

Put $x = e^t$ or $t = \log x$ in the above Eq^n we get.

$D(D-1)y - 4Dy + 6y = \cos 2t$

$[D^2 - 5D + 6]\,y = \cos 2t$

A.E. is $m^2 - 5m + 6 = 0$

$m^2 - 3m - 2m + 6 = 0$

$m(m-3) - 2(m-3) = 0$

$(m-3)(m-2) = 0$

$\therefore$ $m = 2, 3$ The roots are real & distinct.

$\therefore$ $C.F = C_1 e^{2t} + C_2 e^{3t} = C_1 x^2 + C_2 x^3 \longrightarrow (1)$

$P.I = \frac{1}{f(D)} \cdot X = \frac{1}{D^2 - 5D + 6} \cos 2t$ Replace D^2 by -4

$= \frac{1}{(2-5D)} \cdot \cos 2t = \frac{(2+5D)}{(2-5D)(2+5D)} \cdot \cos 2t.$

$= \frac{2\cos 2t - 10 \cancel{\cos}\sin 2t}{(4 - 25D^2)}$ D^2 by -4

$= \frac{1}{104}[2\cos 2t - 10\sin 2t]$

$= \frac{1}{104}[2\cos(2\log x) - 10\sin(2\log x)] \longrightarrow (2)$

$\therefore$ $y = C_1 x^2 + C_2 x^3 + \frac{1}{104}[2\cos(\log x) - 10\sin(2\log x)]$ //

4

a) Given $(D^2-4D+4)y=e^{2x}+\sin x$

A.E is

$m^2-4m+4=0$

$(m-2)^2=0$

$\Rightarrow m=2,2$ The roots are real & repeated.

$C.F = (C_1+C_2x)e^{2x} \longrightarrow (1)$

$P.I = \frac{1}{f(D)} \cdot X = \frac{1}{f(D)} \cdot [e^{2x}+\sin x]$

$= \frac{1}{f(D)} \cdot e^{2x} + \frac{1}{f(D)} \cdot \sin x$

$= \frac{x^2e^{2x}}{2} + \frac{1}{(D^2-4D+4)_{D^2\to -1}} \cdot \sin x$

$= \frac{x^2e^{2x}}{2} + \frac{(3+4D)}{(3+4D)(3-4D)} \sin x$

$= \frac{x^2e^{2x}}{2} + \frac{3\sin x+4\cos x}{(9-16D^2)_{D^2\to -1}}$

$P.I = \frac{x^2e^{2x}}{2} + \frac{1}{25}(3\sin x+4\cos x)$

$\therefore\ y = C.F + P.I.$

i.e $y = (C_1+C_2x)e^{2x} + \frac{x^2e^{2x}}{2} + \frac{1}{25}(3\sin x+4\cos x)$

b) Given $(x+1)^2y''+(x+1)y'+y = 2\sin[\log(x+1)] \longrightarrow (1)$

put $x+1=e^t$ or $t=\log(x+1)$

$\therefore$ Eqn (1) becomes

$D(D-1)y+Dy+y = \sin 2t$

$(D^2+1)y = \sin 2t$

A.E $m^2+1=0$

$\Rightarrow m=\pm i$

P.T.O

$C.F = c_1 \cos t + c_2 \sin t$

$= c_1 \cos[\log(x+1)] + c_2 \sin[\log(x+1)]$

$P.I = \frac{1}{f(D)} . X = \frac{1}{D^2+1} \sin 2t$ Replace D^2 by -4

$P.I = \frac{-1}{3} \sin 2t = \frac{-1}{3} \sin 2[\log(x+1)]$

$\therefore$ The General solution is

$y = C.F + P.I$

ie $y = c_1 \cos[\log(x+1)] + c_2 \sin[\log(x+1)] - \frac{1}{3} \sin[2\log(x+1)]$

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c) Given $\frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{E}{L} \longrightarrow (1)$

Denoting $\lambda^2 = \frac{1}{LC}$ and $\mu = \frac{E}{L}$ in (1) we have

$(D^2 + \lambda^2) q = \mu$

A.E is $m^2 + \lambda^2 = 0 \Rightarrow m^2 = -\lambda^2$

$\therefore m = \pm i\lambda$

$C.F = c_1 \cos \lambda t + c_2 \sin \lambda t \longrightarrow (1)$

$P.I = q_p = \frac{1}{D^2+\lambda^2} \mu = \frac{1}{D^2+\lambda^2} \mu e^{0.t}$ Repace D by 0

$P.I = \frac{\mu}{\lambda^2} \longrightarrow (2)$

$\therefore q(t) = c_1 \cos \lambda t + c_2 \sin \lambda t + \frac{\mu}{\lambda^2} \longrightarrow (3)$

$q'(t) = -\lambda c_1 \sin \lambda t + \lambda c_2 \cos \lambda t.$ But $q(0) = 0$ & $q'(0) = 0$

~~solving~~ $\left.\begin{array}{l} 0 = c_1 + \frac{\mu}{\lambda^2} \\ 0 = \lambda c_2 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = -\frac{\mu}{\lambda^2} \\ c_2 = 0 \end{array}$

$\therefore q(t) = (-\mu/\lambda^2) \cos \lambda t + (\mu/\lambda^2)$

or $q(t) = (\mu/\lambda^2)[1 - \cos \lambda t]$

ie $q(t) = EC\left[1 - \cos \sqrt{\frac{1}{LC}}\, t\right]$

5 (11)

a) Given $\phi(x+y+z,\ x^2+y^2+z^2)=0$

Let $u=x+y+z$ & $v=x^2+y^2+z^2$.

Let $\phi(u,v)=0 \longrightarrow ①$

diff ① partially w.r.t x, y.

$$\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x}=0 \Rightarrow \frac{\partial\phi}{\partial u}\cdot(1+p)+\frac{\partial\phi}{\partial v}\cdot(2x+2zp)$$

$$\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y}=0 \Rightarrow \frac{\partial\phi}{\partial u}(1+q)+\frac{\partial\phi}{\partial v}(2y+2zq)$$

ie $$\frac{\partial\phi}{\partial u}(1+p)=-2(x+zp)\frac{\partial\phi}{\partial v}. \longrightarrow ②$$

$$\frac{\partial\phi}{\partial u}(1+q)=-2(y+zq)\frac{\partial\phi}{\partial v} \longrightarrow ③$$

② ÷ ③ gives $\dfrac{1+p}{1+q}=\dfrac{x+zp}{y+zq}$

$\Rightarrow (1+p)(y+zq)=(x+zp)(1+q)$

$\Rightarrow y+zq+py+\cancel{zpq}=x+zp+xq+\cancel{zpq}$

$x-y=p(z-y)+q(x-z)$ is the required p.d.e.

b) Given $\dfrac{\partial^3 z}{\partial x^2\partial y}=\cos(2x+3y) \longrightarrow ①$

Integrating ① w.r.t x partially treating 'y' as constant.

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\sin(2x+3y)}{2}+f_1(y)$$

Integrating again w.r.t x

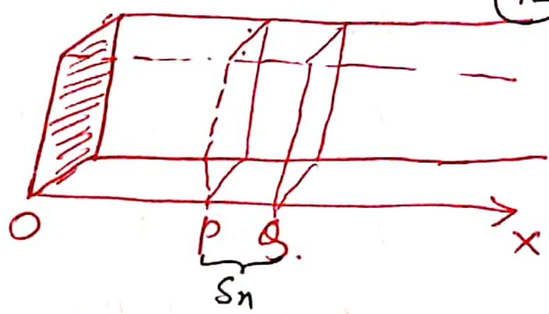
$$\frac{\partial z}{\partial y}=\frac{-\cos(2x+3y)}{4}+xf_1(y)+f_2(y)$$

Integrating w.r.t 'y' partially, treating x as constant

$$z=\frac{-\sin(2x+3y)}{12}+xg_1(y)+g_2(y)+f(x)$$ is the required soln

Where $g_1(y)=\int f_1(y)dy$ & $g_2(y)=\int f_2(y)dy$.

e) Consider a homogeneous ~~uni~~ bar of uniform cross-section α cm^2. ~~t~~ The bar is covered with an impervious material so that heat flows only in parallel & 1^r to the heat beam.



Let ~~the~~ PQ be an small element of length δx. Let $u(x,t)$ be the temperature at any pt. A thin beam of heat is plaurced at 'O', ~~or~~ Let H be the amount of heat flowing through PQ.

ie $\quad H = \rho s \alpha u \delta x$

$$\frac{\partial H}{\partial t} = \rho s \alpha \frac{\partial u}{\partial t} \delta x.$$

The rate of flow of heat at any pt is directly proportional to the rate of change of temp with it's direction.

ie $\quad q \propto \frac{\partial u}{\partial x} \Rightarrow q = -k\alpha \frac{\partial u}{\partial x}$

Let q_1 and q_2 be the rate of flow of heat at P & Q.

$$\therefore \quad q_1 = -k\alpha\left(\frac{\partial u}{\partial x}\right)_x \qquad q_2 = -k\alpha\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}.$$

$$q_1 - q_2 = k\alpha\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

By the law of balance of energy.

$$\rho s \alpha \frac{\partial u}{\partial t}\delta x = k\alpha\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\frac{\partial u}{\partial t} = \frac{k}{\rho s}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

Taking limit as $\delta x \to 0$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{Where } c^2 = \frac{k}{\rho s}$$

P.T.O.

6 a) Given $\frac{\partial^2 z}{\partial x^2} + z = 0$ (13)

Assuming that 'z' is a function of x only.

$$\frac{d^2 z}{dx^2} + z = 0$$

$\therefore$ A.E is $m^2+1=0 \Rightarrow m^2=-1$

$\Rightarrow m = \pm i$. The roots are imaginary.

$\therefore$ General soln is

$$z = C_1 \cos x + C_2 \sin x$$

ie $z = f_1(y)\cos x + f_2(y)\sin x \longrightarrow (1)$

Given $x=0$, $z=e^y$

$\therefore$ Eqn (1) becomes $f_1(y) = e^y$

Also $\frac{\partial z}{\partial x} = -f_1(y)\sin x + f_2(y)\cos x. \longrightarrow (2)$

When $x=0$ $\frac{\partial z}{\partial x} = 1$

$\therefore$ Eqn (2) gives $f_2(y) = 1$

$\therefore$ $z = e^y \cos x + \sin x$ is the required solution

---

b) The auxillary eqn is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

ie $\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \longrightarrow (1)$

Using l, m, n as multipliers

$$\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)} = \frac{ldx+mdy+ndz}{0}$$

$\Rightarrow ldx+mdy+ndz = 0$ Integrating

$lx+my+nz = C_1 \longrightarrow (2)$

Using the multipliers x, y, z.

$$\frac{x\,dx+y\,dy+z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$\Rightarrow x\,dx+y\,dy+z\,dz=0$ Integrating.

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=C_2$$

$$x^2+y^2+z^2=2C_2 \longrightarrow (3)$$

$\therefore$ The general soln is

$$\phi(lx+my+nz,\ x^2+y^2+z^2)=0$$

---

c) $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

Let $u = X(x).T(t)$ be the solution of eqn (1).

$$\therefore \frac{\partial^2}{\partial t^2}(XT) = c^2\frac{\partial^2}{\partial x^2}(XT)$$

$$\Rightarrow XT'' = c^2X''T$$

$$\Rightarrow \frac{T''}{c^2T} = \frac{X''}{X} = K$$

$$\Rightarrow \frac{X''}{X} = k \quad \text{and} \quad \frac{T''}{c^2T} = k$$

ie $\frac{d^2X}{dx^2} - kX = 0$ and $\frac{d^2T}{dt^2} = kc^2T$

ie $(D^2-k)X=0$ and $(D^2-c^2k)T=0$

A.E is $D^2-k=0$ and $D^2-c^2k=0$

Case i: When k is +ve ie $k=p^2$ $D=\pm p$ and $D=\pm cp$.

$\therefore X = C_1e^{-px}+C_2e^{px}$ and $T = C_3e^{pct}+C_4e^{-pct}$.

Case ii: $k=-p^2 \Rightarrow$ $D=\pm pi$ & $D=\pm cpi$

$X = C_5\cos px + C_6\sin px$ and $T = C_7\cos cpt + C_8\sin cpt$

Case iii: $k=0 \Rightarrow$ $D=0,0$ & $T_1=0,0$

$\therefore X = C_9+C_{10}x$ and $T = T = C_{11}+C_{12}x$.

Since we are dealing with vibration of string which θ is periodic in nature. Therefore the solution must contain trignometric terms.

$\therefore$ $y = [c_1 \cos px + c_2 \sin px][c_3 \cos cpt + c_4 \sin cpt]$

is the required soln.

7

a) Let $u_n = \frac{(n+1)^n}{n^{n+1}} \cdot x^n$

$$(u_n)^{1/n} = \frac{(n+1)x}{n \cdot n^{1/n}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{(n+1)x}{n \cdot (n)^{1/n}}$$

$$= \lim_{n\to\infty} \frac{n(1+1/n)x}{n(n)^{1/n}}$$

$$= \lim_{n\to\infty} \frac{(1+1/n)x}{n^{1/n}} = \frac{(1+0)x}{1} = x$$

Therefore by Cauchy's root test

Σu_n is convergent if $x<1$, divergent if $x>1$ and the test fails if $x=1$.

b) We have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$

putting $n = 1/2$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(1/2+r+1)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2r}$$

$$= \left(\frac{x}{2}\right)^{1/2} \left[\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(\frac{1}{2}+r+1)} \cdot \left(\frac{x}{2}\right)^{2r} \right]$$

$$= \left(\frac{x}{2}\right)^{1/2} \left[\frac{1}{\Gamma(3/2)} - \frac{1}{\Gamma(5/2)} \cdot \left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma(7/2) \cdot 2!} \left(\frac{x}{2}\right)^4 - \cdots \right]$$

(16)

$$= \frac{\sqrt{x}}{\sqrt{2}}\left[\frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}}\cdot\frac{1}{\sqrt{\pi}}\left[2 - \frac{1}{3}\cdot x^2 + \frac{1}{15}x^4 - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{2}{x}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\sin x.$$

ie $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$.

c) WKT. $P_1(x) = x$

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

$$\therefore x^3+2x^2-x+1 = \frac{2}{5}P_3(x) + \frac{1}{3}P_0(x) + 2\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] - P_1(x) + P_0(x)$$

$$= \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) + \left(\frac{3}{5}-1\right)P_1(x) + \left(\frac{2}{3}+1\right)P_0(x)$$

$$= \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) + \frac{5}{3}P_0(x)$$

$\therefore a = 2/5,\ b = 4/3,\ c = -2/5,\ d = 5/3$

8 a) Let $u_n = \frac{x^n}{n(n+1)}$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{x^n} = \frac{n}{n+2}\cdot x$$

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{n}{n+2}x$$

$$= \lim_{n\to\infty}\left[\frac{1}{(1+2/n)^n}\right] = x$$

$\therefore$ By D'Alemberts ratio test.

Σu_n is convergent if $x<1$

Σu_n is divergent if $x>1$

& the test fails if $x=1$

b) We have Legendres differential equation ~~is given~~

$$(1-x^2)\cdot y'' - 2xy' + n(n+1)y = 0 \longrightarrow (1)$$

Let $y = \sum_{r=0}^{\infty} a_r x^r$

$$\frac{dy}{dx} = \sum_0^{\infty} a_r r x^{r-1}, \quad \frac{d^2y}{dx^2} = \sum_0^{\infty} a_r r(r-1) x^{r-2}$$

$\therefore$ Eq^n (1) becomes

$$(1-x^2)\sum_0^{\infty} a_r r(r-1)x^{r-2} - 2x\sum_0^{\infty} a_r r x^{r-1} + n(n+1)\sum_0^{\infty} a_r x^r = 0$$

$$\text{ie } \sum_0^{\infty} a_r r(r-1)x^{r-2} - \sum_0^{\infty} a_r r(r-1)x^r - \sum_0^{\infty} 2a_r r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

Equating to zero the coefficients of various powers of x to zero

We first equate the coeff of x^{-2} and x^{-1} to zero

coeff of x^{-2}: $a_0(0)(-1) = 0 \Rightarrow a_0 \neq 0$

coeff of x^{-1}: $a_1(1)(0) = 0 \Rightarrow a_1 \neq 0$

Now equating the coefficient of x^r to zero $(r \geq 0)$

$$a_{r+2}(r+2)(r+1) - a_r r(r-1) - 2a_r r + n(n+1)a_r = 0$$

$$\text{or } a_{r+2} = \frac{-[n(n+1) - r^2 - r]}{(r+2)(r+1)}\cdot a_r \longrightarrow (2)$$

putting $r = 0, 1, 2, 3, \dots$ in (2) we get.

$$a_2 = \frac{-n(n+1)}{2}\cdot a_0; \quad a_3 = \frac{-(n^2+n-2)}{6}a_1 = -\frac{(n-1)(n+2)}{6}a_1$$

$$a_4 = \frac{-(n^2+n-6)}{12}a_2 = \frac{n(n+1)(n-2)(n+3)}{24}a_0$$

P.T.O

$$a_5 = \frac{-(n^2+n-12)}{20} a_3 = \frac{(n-1)(n+2)(n-3)(n+4)}{120} a_1 \text{ and so on}$$ (18)

$$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

ie $$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 - \cdots\right] +$$

$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}x^5 - \cdots\right]$$

Let $u(x)$ & $v(x)$ represents two infinite series

we have

$$y = a_0 u(x) + a_1 v(x)$$

This is the series solution of Legendre differential equation

---

c) we have

$$x^4 = \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}P_0$$

$$x^3 = \frac{2}{5}P_3 + \frac{3}{5}P_1, \quad x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

$P_1(x) = x$, & $P_0(x) = 1$

$$\therefore x^4 - 3x^2 + x = \left(\frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}P_0\right) - 3\left(\frac{2}{3}P_2 + \frac{1}{3}P_0\right) + P_1(x)$$

$$= \frac{8}{35}P_4 + \left(\frac{4}{7} - 2\right)P_2 + \left(\frac{1}{5} - 1\right)P_0 + P_1$$

$$= \frac{8}{35}P_4 - \frac{10}{7}P_2 + P_1 - \frac{4}{5}P_0$$

$$\therefore x^4 - 3x^2 + x = \frac{8}{35}P_4 - \frac{10}{7}P_2 + P_1 - \frac{4}{5}P_0 \quad //$$

9.

(19)

a) $f(x) = x\sin x + \cos x \qquad x_0 = \pi$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x) = x\cos x + \sin x - \sin x$$

$$f'(x) = x\cos x$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{\cos\pi + \pi\sin\pi}{\pi\cos\pi} = 2.82328$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.82328 - \frac{2.82328 \cdot \sin(2.82328) + \cos(2.82328)}{(2.82328)\cos(2.82328)}$$

$$x_2 = 2.79859$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.79859 - \frac{2.79859\sin(2.79859) + \cos(2.79859)}{(2.79859)\cos(2.79859)}$$

$$x_3 = 2.7983$$

$\therefore$ The roots is $x = 2.7983$.

b) Let $y = f(x)$ represents the number of students having less than or equal to x marks.

Preparing the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 40$	$y_0 = 31$	$\Delta y_0 = 42$	$\Delta^2 y_0 = 9$	$\Delta^3 y_0$ -25	$\Delta^4 y_0$ 37
50	73	52	-16	12	
60	124	35	-4		
70	159	31			
80	190				

(a) Let $x_p = 45 = x_0 + ph$

$$p = \frac{x_p - x_0}{h} = \frac{45-40}{10} = 0.5$$

$$\boxed{P = 0.5}$$

By NFIF

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$y(45) = 31 + (0.5)\times 42 + \frac{0.5(0.5-1)}{2!}\times 9 + \frac{0.5(0.5-1)(0.5-2)}{3!}\times(-25) + + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!} + \cdots$$

$$y(45) = 47.85 \approx 48$$

∴ The number of students who score less than 45 marks is 48.

b) The number of students between 40 & 45 marks

$= 48 - 31 = 17$ (students)

c) $h = \frac{b-a}{n} = \frac{1-0}{6} = 1/6$

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x) = \frac{1}{1+x^2}$	0 y_0	0.1621 y_1	0.3 y_2	0.4 y_3	0.4615 y_4	0.4918 y_5	0.5 y_6

By Simpson's 3/8 rule.

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + y_6\right]$$

$$= \frac{3}{48}\left[0 + (3\times 0.1621) + (3\times 0.3) + (2\times 0.4) + (3\times 0.4615) + + (3\times 0.4918) + 0.5\right]$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = 0.7854$$

10 a) $f(x) = x\log x - 1.2$ (2.1)

$f(2) = -0.5979$ & $f(3) = 0.2313$

$\therefore$ The roots lies in the interval $(2,3) = (a,b)$

$\therefore$ By Regula falsi method

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2\times(0.2313) - (3)(-0.5979)}{0.2313 + 0.5979}$$

$x_1 = 2.7210$

$f(x_1) = 1.5237 > 0$

$\therefore$ The new interval is $(2, 2.7210)$

$$x_2 = \frac{2\times 1.5237 - (2.7210)(-0.5979)}{1.5237 + 0.5979}$$

$x_2 = 2.74021$

$f(x_2) = 1.5622 > 0$

$$x_3 = \frac{2\times 1.5622 - (2.7402)(-0.5979)}{1.5622 + 0.5979}$$

$x_3 = 2.74024$

$f(x_3) = 1.5622 > 0$

$$x_4 = \frac{2\times 1.5622 - (2.74024)(-0.5979)}{1.5622 + 0.5979}$$

$x_4 = 2.74063$

$f(x_4) = 1.5630$

$$x_5 = \frac{2\times 1.5630 - (2.74063)(-0.5979)}{1.5630 + 0.5979}$$

$x_5 = 2.74063.$

$\therefore$ The real root is $\alpha = 2.7406$ //

(22)

b) By Lagranges Interpolation formula.

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 +$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 + \cdots$$

$$f(4) = \frac{(4-1)(4-2)(4-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(4-0)(4-2)(4-5)}{(1-0)(1-2)(1-5)} \times 3$$

$$+ \frac{(4-0)(4-1)(4-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(4-0)(4-1)(4-2)}{(5-0)(5-1)(5-2)} \times (147)$$

$$f(4) = 78$$

c) Let $h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$ and $n=6$

x	4	4.2	4.4	4.6	4.8	5	5.2
$y = \log_e x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By weddles rule

$$\int_a^b y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$= \frac{3\times(0.2)}{10}\left[1.3863 + (5\times 1.4351) + 1.4816 + (6\times 1.5261) + 1.5686 + (5\times 1.6094) + (1.6487)\right]$$

$$\int_4^{5.2} \log_e x = 1.8279$$

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