MODEL QUESTION PAPER - 2

Complex Analysis, Probability & Statistical Methods [18MAT41]

Time: 3 hours **Max Marks: 100**

Note: Answer any FIVE full questions, choosing ONE full question from each module.

Module – 1

1. a) Show that $\omega = f(z) = z + e^z$ is analytic and hence find $\frac{dw}{dz}$. (6)

b) Derive Cauchy-Reimann equation in cartesian form. (7)

c) Find the analytic function $f(z) = u + iv$ given $u - v = e^x(cosy - siny)$ (7)

OR

2. a) If u and v are harmonic functions, show that $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is analytic. (6)

b) If $f(z)$ is analytic, show that $\left[\left({\partial^2}/{\partial x^2}\right) + \left({\partial^2}/{\partial y^2}\right)\right] |f(z)|^2 = 4|f'(z)|^2$. (7)

c) Determine the analytic function $f(z)$ whose imaginary part is $\left(r - \frac{k^2}{r}\right) sin\theta$, $r \neq 0$. Hence, find the real part $f(z)$. (7)

Module – 2

3 a) State and prove Cauchy's theorem. (6)

b) Discuss the transformation $w = z^2$. (7)

c) Find the bilinear transformation which maps the points $z = -1, 0, 1$ into the points $\omega = 0, i, 3i$ respectively. (7)

OR

4. a) Verify Cauchy's theorem for $f(z) = ze^{-z}$ over the unit circle with origin as the center. (7)

b) Evaluate $\int_C \frac{z}{(z^2+1)(z^2-9)} dz$ where C is the circle $C: |z| = 2$. (6)

c) Evaluate $\int_C |z|^2 dz$ where C is a square with the vertices (0.0), (1,0), (1,1), (0,1). (7)

Module – 3

5. a) A random variable has the following probability function for various values of x:

X	-3	-2	-1	0	1	2	3
$P(X)$	k	$2k$	$3k$	$4k$	$3k$	$2k$	k

Find the value of k, mean and variance.

b) After the appointment of a new sales manager, the sales in a two-wheeler showroom is exponentially distributed with mean 4. If two days are selected at random, what is the probability that (i) on both days the sales is over 5 units, (ii) the sales is over 5 units on atleast one of the two days. (6)

c) In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and SD of 60 hours. If a firm purchases 2500 bulbs find the number of bulbs that are likely to last for i) more than 2100 hours ii) less than 1950 hours iii) between 1900 to 2100 hours (7)

OR

6. a) The probability density function of a random variable $X(= x)$ is (6)

$$f(x) = \begin{cases} kx, & 0 \le x \le 5 \\ 0, & elsewhere \end{cases}$$

Evaluate the value of k and find $P(1 \le x \le 3)$ and (iii) $P(x \le 3)$.

b) While forming 36-digit binary numbers a malfunctioning digital computer has found to form a digit incorrectly about 1 times in 1000. Assuming the errors in forming different digits are independent, find the probability of having (i) zero (ii) more than one (iii) less than five incorrect digits in a given 36-digit number. (7)

c) The weekly wages of 1000 workers in an electric company are normally distributed with mean of Rs. 700 and standard deviation of Rs. 50. Estimate the number of workers whose weekly wages will be (i) between Rs. 700 and Rs. 720 (ii) more than Rs. 750. (7)

Module – 4

7. a) An experiment on lifetime (t) of cutting tool at different cutting speeds v units are given (6)

Speed (v)	350	400	500	600
Lifetime (t)	61	26	7	2.6

Fit a relation of the form $v = at^b$.

b) Fit a best fitting parabola in the form $y = ax^2 + bx + c$ for the following data (7)

y	1	1.5	2	2.5	3	3.5	4
x	1.1	1.3	1.6	2	2.7	3.4	4.1

c) Ten competitors in music contestant are ranked by 3 judges A, B, C in the following order. Use the rank correlation coefficient to decide which pair of judges have the most approach to common taste of music. (7)

A	1	6	5	10	3	2	4	9	7	8
B	3	5	8	4	7	10	2	1	6	9
C	6	4	9	8	1	2	3	10	5	7

OR

8. a) $8x - 10y + 66 = 0$, $40x - 18y = 214$ are the two regression lines. Find the mean of x's, y's and the correlation coefficient. Find σ_y if $\sigma_x = 3$. (6)

b) Find the equation of the best fitting straight line for the following data and hence estimate the value of the dependent variable corresponding to the value 30 of the independent variable (7)

x	5	10	15	20	25
y	16	19	23	26	30

c) Obtain the lines of regression and hence find the coefficient of correlation for the data (7)

x	1	3	4	2	5	8	9	10	13	15
y	8	6	10	8	12	16	16	10	32	32

Module – 5

9. a) The manufacturer of a certain medicine claimed that it was 90% effective in relieving an allergy for a period of 8 hours. In a sample of 200 persons who had the allergy, the medicine provided relief for 160 persons. Determine whether the manufacturer's claim is legitimate at 0.01 level of significance. (6)

b) A joint probability distribution is given by the following table
Determine (i) Marginal distributions of X and Y (ii) $COV(X, Y)$ (c) correlation of X and Y. (7)

X \ Y	3	4	5
2	1/6	1/6	1/6
5	1/12	1/12	1/12
7	1/12	1/12	1/12

c) It has been found from experience that the mean breaking strength of a particular brand of thread is 275.6 gms with standard deviation of 39.7 gms. Recently a sample of 36 pieces of thread showed a mean breaking strength of 253.2 gms. Can one conclude at a significance level of (a) 0.05 (b) 0.01 that the thread has become inferior? (7)

OR

10. a) A coin is tossed 1000 times and head turns up 540 times. Decide on the hypothesis that the coin is unbiased. (6)

b) A sample of 100 tyres is taken from a lot. The mean life of tyres is found to be 39350 kilometers with a standard deviation of 3260.Can it be considered as a true random sample from a population with mean life of 40,000 kms. Use 0.05 level of significance. Establish 99% confidence limits with in which the mean life of tyres expected to lie. (Given that $Z_{0.05} = 1.96$, $Z_{0.01} = 2.58$) (7)

c) Fit a Poisson's distribution to the following data and test the goodness of fit at 5% level of significance, given that $\chi^2_{0.05} = 7.815$ for 3 d.f. (7)

No. of heads	0	1	2	3	4
Frequency	5	29	36	25	5

***********END***********

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MODEL QUESTION PAPER

COMPLEX ANALYSIS, PROBABILITY AND STATISTICAL METHODS (18MAT41)

Module-1

1 (a) Show that $\omega = z + e^z$ is analytic, and hence find $d\omega/dz$.

Sol^n: Given, $\omega = z + e^z$

$$u + iv = (x+iy) + e^{x+iy}$$

$$= (x+iy) + e^x \cdot e^{iy}$$

$$= (x+iy) + e^x(\cos y + i \sin y)$$

$$= (x + e^x \cos y) + i(y + e^x \sin y)$$

$\therefore u = x + e^x \cos y$ $\qquad$ $v = e^x \sin y + y$

$u_x = 1 + e^x \cos y$ $\qquad$ $v_x = e^x \sin y$

$u_y = -e^x \sin y$ $\qquad$ $v_y = e^x \cos y + 1$

The Cauchy Reimann equations in cartesian form are $u_x = v_y$ and $v_x = -u_y$ are satisfied

i.e. $u_x = 1 + e^x \cos y = v_y$ & $v_x = e^x \sin y = -u_y$

Thus, $\omega = z + e^z$ is analytic.

Also, $\dfrac{d\omega}{dz} = f'(z) = u_x + iv_x$

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$$\frac{dw}{dz} = (1+e^x\cos y) + i(e^x \sin y)$$

$$= 1+e^x(\cos y + i\sin y) = 1+e^x . e^{iy}$$

$$= 1+e^{x+iy} = 1+e^z \quad \because z = x+iy$$

Thus, $\frac{dw}{dz} = 1+e^z$

(b) Derive Cauchy-Reimann equations in Cartesian form.

Proof: Statement: The necessary conditions that the function $w = f(z) = u(x,y) + i\,v(x,y)$ be analytic at any point $z = x+iy$, is that there exists four continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the equations : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Proof: Let, $f(z)$ be analytic at a point $z = x+iy$ and by the definition

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ exists & is unique.

In the cartesian form, $f(z) = u(x,y) + i\,v(x,y)$ and let δz be the increment in z corresponding to the increments $\delta x, \delta y$ in x, y.

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$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + i v(x+\delta x, y+\delta y)] - [u(x,y) + i v(x,y)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x) y+\delta y) - u(x,y)]}{\delta z} + i \lim_{\delta z \to 0} \frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta z} \longrightarrow (1)$$

Now, $\delta z = (z+\delta z) - z$, where $z = x+iy$

$\delta z = [(x+\delta x) + i(y+\delta y)] - [x+iy]$

i.e. $\delta z = \delta x + i\delta y$

Since, δz tends to zero, we have the following two possibilities.

<u>Case (i)</u>: Let, $\delta y = 0$ so that $\delta z = \delta x$ and $\delta z \to 0$ imply $\delta x \to 0$.

Eqn (1) becomes,

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

These limits from the basic definitions are the partial derivatives of u & v w.r.t. 'x'

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Therefore, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow (2)$

Case (ii): Let $\delta x = 0$, so that $\delta z = i\delta y$ and $\delta z \to 0$ imply $i\delta y \to 0$ or $\delta y \to 0$.

Now, eqn (1) becomes,

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i\lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

But, $1/i = i/i^2 = i/-1 = -i$ & hence we have

$$f'(z) = \lim_{\delta y \to 0} -i \cdot \frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \longrightarrow (3)$$

Equating the RHS of (2) & (3),

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

By equating the real & imaginary parts we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Thus, we have established Cauchy-Reimann equations.

Mmk

(c) Find the analytic function $f(z) = u+iv$, given $u-v = e^x(\cos y - \sin y)$.

Solⁿ: Given, $u-v = e^x(\cos y - \sin y)$

Differentiating w.r.t. x & y partially,

$$u_x - v_x = e^x(\cos y - \sin y) \longrightarrow (1)$$

and $u_y - v_y = e^x(-\sin y - \cos y) \longrightarrow (2)$

Using Cauchy-Reimann equations for the LHS of eqn (2) in the form $u_y = -v_x$ & $v_y = u_x$

we have

$$-v_x - u_x = e^x(-\sin y - \cos y)$$

or $u_x + v_x = e^x(\sin y + \cos y) \longrightarrow (3)$

Solving eqns (1) & (3) simultaneously for u_x & v_x

$(1)+(3)$: $2u_x = 2e^x\cos y$ $\therefore u_x = e^x\cos y$

$(1)-(3)$: $-2v_x = -2e^x\sin y$ $\therefore v_x = e^x\sin y$

$f'(z) = u_x + iv_x$

$f'(z) = e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y)$

$= e^x.e^{iy} = e^{x+iy} = e^z$

$\therefore f(z) = e^z + c$ (integrating wrt z)

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2a. If u & v are harmonic functions, show that $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is analytic.

Sol^n: Let, $P = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$, $Q = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

To prove that $P+iQ$ is analytic, we show that C-R equations in the form $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$ and $\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}$ are satisfied.

Consider, $\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 v}{\partial y^2}$$

$$= -\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0,$$ since v is harmonic.

$\therefore \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0$ or $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$

Also, consider $\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$

$$= \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}$$

$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, since u is harmonic.

$\therefore \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0$ or $\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}$

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Thus, $P+iQ$ is analytic.

2b. If, $f(z)$ is analytic, show that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] |f(z)|^2 = 4|f'(z)|^2$$

Soln: Let, $f(z) = u + iv$ be analytic.

$\therefore |f(z)| = \sqrt{u^2+v^2}$ or $|f(z)|^2 = u^2+v^2 = \phi$

To prove that, $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\phi = 4|f'(z)|^2$

That is to prove that, $\phi_{xx} + \phi_{yy} = 4|f'(z)|^2$

Consider, $\phi = u^2+v^2$ and differentiate w.r.t. 'x' partially.

$$\phi_x = 2uu_x + 2vv_x = 2[uu_x + vv_x]$$

Differentiating again w.r.t. 'x' we get,

$$\phi_{xx} = 2[uu_{xx} + u_x^2 + vv_{xx} + v_x^2] \longrightarrow (1)$$

$$\phi_{yy} = 2[uu_{yy} + u_y^2 + \cancel{v}\, v_{yy} + v_y^2] \longrightarrow (2)$$

Adding (1) and (2) we have,

$$\phi_{xx} + \phi_{yy} = 2[u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2] \longrightarrow (3)$$

Since, $f(z) = u+iv$ is analytic, u & v are harmonic.

Hence, $u_{xx}+u_{yy} = 0$, $v_{xx}+v_{yy} = 0$.

Further, we have C-R equations $v_y = u_x$ and $u_y = -v_x$

Using these results in the RHS of (3), we have

$$\phi_{xx} + \phi_{yy} = 2\left[u.0 + v.0 + u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2\right]$$

i.e. $\phi_{xx} + \phi_{yy} = 2\left[2u_x^2 + 2v_x^2\right]$

$$= 4\left[u_x^2 + v_x^2\right] \longrightarrow (4)$$

But, $f'(z) = u_x + iv_x$

$\therefore |f'(z)| = \sqrt{u_x^2 + v_x^2} \Rightarrow |f'(z)|^2 = u_x^2 + v_x^2$

Using this in the RHS of (4) we have

$$\underline{\underline{\phi_{xx} + \phi_{yy} = 4|f'(z)|^2}}$$

2c) Determine the analytic function $f(z)$ whose imaginary part is $\left(r - \frac{k^2}{r}\right)\sin\theta$, $r \neq 0$. Hence, find the real part ~~if~~ of $f(z)$.

<u>Sol</u>: Let, $v = \left(r - \frac{k^2}{r}\right)\sin\theta$

Differentiating wrt 'r' & 'θ' is partially,

$$v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta, \quad v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$$

Consider, $f'(z) = e^{-i\theta}(u_r + iv_r)$.

But, $u_r = \frac{1}{r}v_\theta$ (Cauchy Reimann equation)

$$f'(z) = e^{-i\theta}\left(\frac{1}{r}v_\theta + iv_r\right)$$

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$$f'(z) = e^{-i\theta}\left[\left(1-\frac{k^2}{r^2}\right)\cos\theta + i\left(1+\frac{k^2}{r}\right)\sin\theta\right] \quad \longrightarrow (1)$$

putting $r = z$ and $\theta = 0$, we have

$$f'(z) = 1 - \frac{k^2}{z^2}$$

Integrating w.r.t. 'z',

$$f(z) = \int \left(1 - \frac{k^2}{z^2}\right) dz + c$$

Thus, $$f(z) = \left(z + \frac{k^2}{z}\right) + c$$

Now, let us find $u(r,\theta)$ from $f(z)$ by putting $z = re^{i\theta}$.

$$u + iv = (re^{i\theta}) + \frac{k^2}{re^{i\theta}} = r(\cos\theta + i\sin\theta) + \frac{k^2}{r}(\cos\theta - i\sin\theta)$$

$$u + iv = \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$$

Thus, the required real part

$$u = \left(r + \frac{k^2}{r}\right)\cos\theta$$

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MODULE-2

3a) State and prove Cauchy's Theorem.

Statement: If $f(z)$ is analytic at all points inside and on a simple closed curve C then $\int_C f(z)\,dz = 0$.

Proof: Let, $f(z) = u+iv$

Then, $\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Applying Green's theorem in a plane stating that if $M(x,y)$ and $N(x,y)$ are two real valued functions having continuous first order partial derivatives in a region R bounded by the curve C then

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Applying this theorem to the two line integrals in the RHS of (1) we obtain,

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$
$$+ i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Since $f(z)$ is analytic, we have Cauchy-Reimann equations : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ & hence we have

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$$\int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Thus, we get $\int_C f(z)\,dz = 0$

This proves Cauchy's Theorem.

3b. Discuss the transformation $w = z^2$.

Solⁿ: Consider, $w = z^2$

$u + iv = (x+iy)^2$ or $u+iv = (x^2 - y^2) + i(2xy)$

$\therefore u = x^2 - y^2$ and $v = 2xy$ $\longrightarrow$ (1)

Case-1: Let us consider $x = c_1$, c_1 is constant

The set of equations (1) become

$u = c_1^2 - y^2$; $v = 2c_1 y$

Now, $y = v/2c_1$ and substituting this in u we get

$u = c_1^2 - (v^2/4c_1^2)$ or $v^2/4c_1^2 = c_1^2 - u$

or $v^2 = -4c_1^2(u - c_1^2)$

This is a parabola in the w-plane symmetrical about the real axis with its vertex at $(c_1^2, 0)$ and focus at the origin. It may be noted that the line $x = -c_1$ is also transformed into the same parabola.

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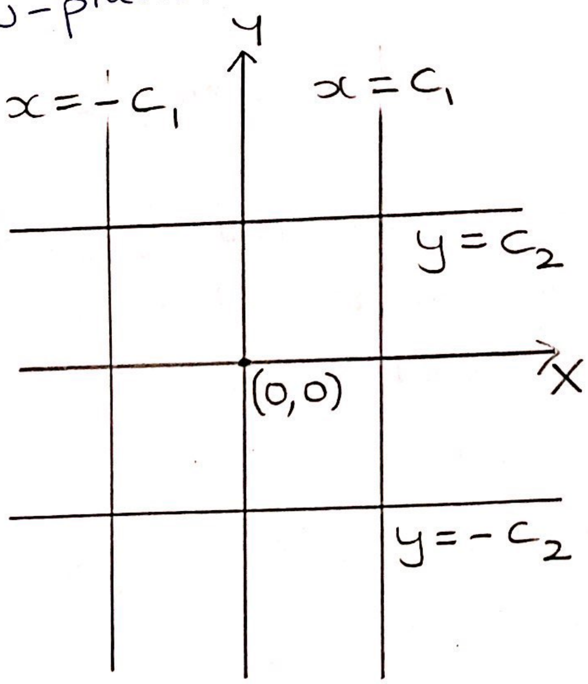
Case 2: Let us consider $y = c_2$, c_2 is constant.
The set of equations (1) become $u = x^2 - c_2^2$,

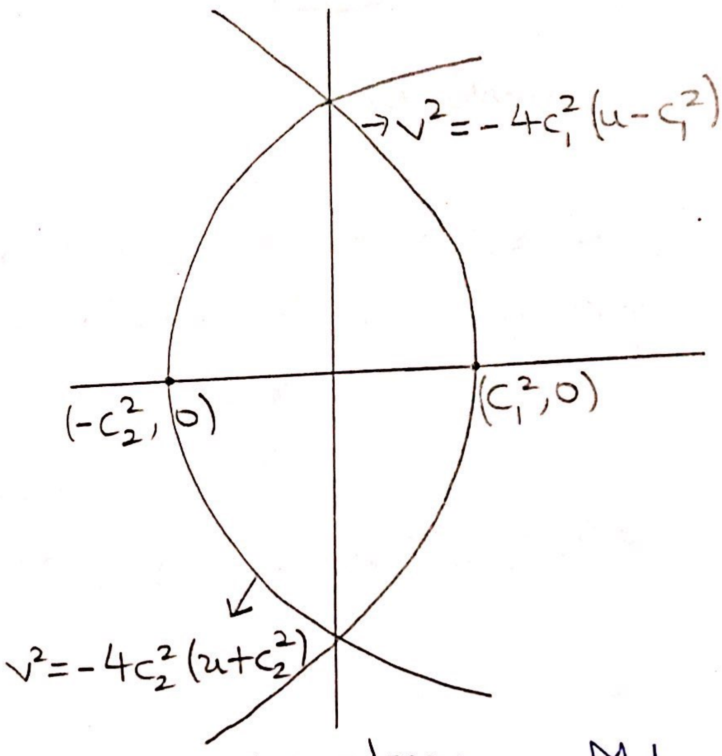
$$v = 2xc_2$$

$\Rightarrow x = v/2c_2$ and substituting this in 'u'
we get $u = (v^2/4c_2^2) - c_2^2$
or $v^2/4c_2^2 = u + c_2^2$ or $v^2 = 4c_2^2(u + c_2^2)$

This is also a parabola in the ω-plane symmetrical about the real axis whose vertices is at $(-c_2^2, 0)$ and focus at the origin. Also, the line $y = -c_2$ is transformed into the same parabola. Hence, from these two cases we conclude that the straight lines parallel to the co-ordinate axes in the z-plane map onto parabolas in the ω-plane.



z-plane



ω-plane

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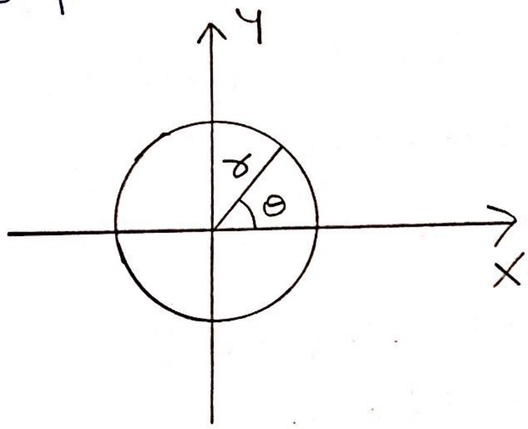
Case-3 : Let us consider a circle with centre origin and radius 'r' in the z-plane.

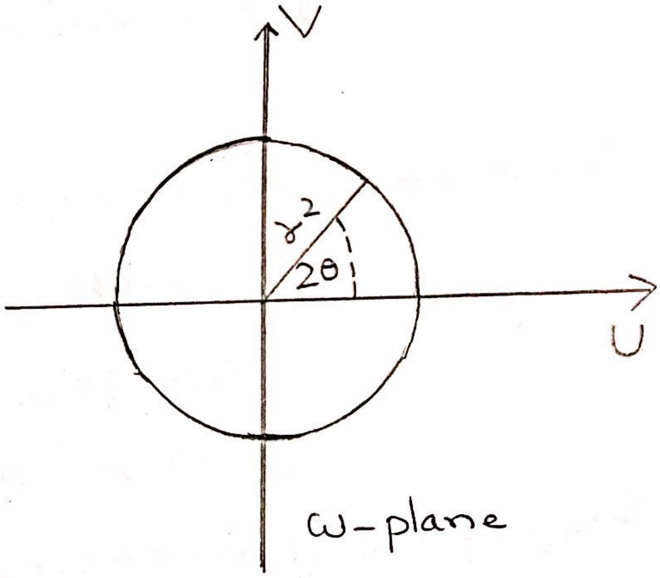
i.e. $|z| = r$ $\therefore z = re^{i\theta}$. Hence, $w = z^2 = (re^{i\theta})^2$

$$w = r^2 e^{2i\theta} = Re^{i\phi} \text{ so that } R = r^2 \text{ and } \phi = 2\theta.$$

This is also a circle in the w-plane having radius r^2 and subtending an angle 2θ at the origin.
Hence, we conclude that a circle with centre origin and radius 'r' in the z-plane maps onto a circle with centre origin and radius r^2 in the w-plane.



z-plane



w-plane

Case-4: Let us consider a circle with centre 'a' and radius 'r' in the z-plane whose equation in the complex form is $|z-a| = r$.

$$z - a = re^{i\theta} \quad \text{or} \quad z = a + re^{i\theta}$$

Hence, $w = z^2 = (a + re^{i\theta})^2 = a^2 + 2are^{i\theta} + r^2 e^{2i\theta}$

$$w - a^2 = 2are^{i\theta} + r^2 e^{2i\theta}$$

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Adding r^2 on both sides,

$$w - a^2 + r^2 = 2are^{i\theta} + r^2 + r^2 e^{2i\theta}$$

$$= 2are^{i\theta} + r^2(1 + e^{2i\theta})$$

$$w - (a^2 - r^2) = 2re^{i\theta}\left[a + \frac{r}{2}(e^{-i\theta} + e^{i\theta})\right]$$

$$w - (a^2 - r^2) = 2re^{i\theta}(a + r\cos\theta)$$

[using $2\cos\theta = e^{-i\theta} + e^{i\theta}$]

Suppose, $w - (a^2 - r^2) = Re^{i\phi}$ then this eqn becomes $Re^{i\phi} = 2re^{i\theta}(a + r\cos\theta)$ so that pole in the w-plane is at the point $(a^2 - r^2)$.

Now, $R(\cos\phi + i\sin\phi) = 2r(a + r\cos\theta)(\cos\theta + i\sin\theta)$

$$R\cos\phi = 2r(a + r\cos\theta)\cos\theta$$

$$R\sin\phi = 2r(a + r\cos\theta)\sin\theta$$

Squaring and adding these we have,

$$R^2(\cos^2\phi + \sin^2\phi) = [2r(a + r\cos\theta)]^2(\cos^2\theta + \sin^2\theta)$$

$$R^2 = [2r(a + r\cos\theta)]^2$$

$$\Rightarrow R = 2r(a + r\cos\theta) \longrightarrow (1)$$

Also, $\frac{R\sin\phi}{R\cos\phi} = \tan\theta$ or $\tan\phi = \tan\theta \Rightarrow \theta = \phi$

Using this in eqn (1), $R = 2r(a + r\cos\phi) \longrightarrow (2)$

The curve (3) is called a Limacon. Hence, we conclude that the circle maps onto a Limacon in the w-plane.

Mwy

3c. Find the bilinear transformation which maps the points $z=-1,0,1$ into the points $w=0, i, 3i$ respectively.

Solⁿ: Let, $w=\dfrac{az+b}{cz+d}$ be the required linear transformation.

$z_1=-1,\ z_2=0,\ z_3=1$ & $w_1=0,\ w_2=i,\ w_3=3i$

$z_1=-1,\ w_1=0$; $0=\dfrac{-a+b}{-c+d}$

$-a+b=0 \longrightarrow (1)$

$z_2=0,\ w_2=i$; $i=\dfrac{0+b}{0+d}$

i.e. $b-di=0 \longrightarrow (2)$

$z_3=1,\ w_3=3i$; $3i=\dfrac{a+b}{c+d}$

i.e. $a+b-3ic-3id=0 \longrightarrow (3)$

(1) − (2) gives, $-a+di=0 \longrightarrow (4)$

Solving (2) & (4), $0a+1b-id=0$

$-1a+0b+id=0$

Applying the rule of cross multiplication,

$$\frac{a}{\begin{vmatrix}1 & -i\\ 0 & i\end{vmatrix}}=\frac{-b}{\begin{vmatrix}0 & -i\\ -1 & i\end{vmatrix}}=\frac{d}{\begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix}}$$

$$\frac{a}{i}=\frac{-b}{-i}=\frac{d}{1} \Rightarrow a=i,\ b=i,\ d=1$$

MK

With these values (3) becomes $i+1-3ic-3i=0$.

Substituting the values of a, b, c, d the assumed bilinear transformation becomes,

$$\omega = \frac{iz+i}{-\frac{z}{3}+1} = \frac{3i(z+1)}{-z+3}$$

This can also be written in the form

$$\omega = \frac{3i(z+1)}{i(iz-3i)} = \frac{3z+3}{iz-3i}$$

OR

4a. Verify Cauchy's theorem for $f(z) = ze^{-z}$ over the unit circle with origin as the centre.

Solⁿ: We know that $|z-a| = r$ or $z-a = re^{i\theta}$

Since, $|z| = 1 \Rightarrow z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$$\Rightarrow dz = ie^{i\theta}\, d\theta$$

$$\int_C ze^{-z}\, dz = \int_{\theta=0}^{2\pi} e^{i\theta} . e^{-i\theta} e^{-e^{i\theta}}\, i\, d\theta = i\int_{\theta=0} e^{2i\theta} e^{-e^{i\theta}}\, d\theta$$

put, $e^{i\theta} = t$, $\therefore e^{i\theta} i\, d\theta = dt$ or $d\theta = \frac{dt}{it}$

when $\theta = 0$, $t = e^0 = 1$, when $\theta = 2\pi$, $t = e^{2\pi i}$
$= \cos 2\pi + i \sin 2\pi = 1$

$$\int_C ze^{-z}\, dz = i\int_{t=1}^{1} t^2 e^{-t} \frac{dt}{it} = \int_1^1 te^{-t}\, dt = 0$$

Thus, $\int_C ze^{-z}\, dz = 0$. Hence, the theorem is verified.

4b. Evaluate $\int_C \frac{z}{(z^2+1)(z^2-9)} dz$ where $C: |z| = 2$.

Soln: $|z| = 2$ is a circle with centre at $z = 0$ & radius 2.

$z^2+1 = 0 \Rightarrow z = \pm i$ which lies inside the circle $|z| = 2$.

$z^2-9 = 0 \Rightarrow z = \pm 3$ which lies outside the circle $|z| = 2$.

Let, $f(z) = \frac{z}{z^2-9}$ which is analytic within and on the circle $|z| = 2$.

Now, $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} =$

$$\frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$$1 = A(z-i) + B(z+i)$$

put, $z = i \Rightarrow 1 = B(i+i) = 2Bi \Rightarrow B = 1/2i$

put, $z = -i \Rightarrow 1 = A(-i-i) = -2Ai \Rightarrow A = -1/2i$

$$\int_C \frac{z/(z^2-9)}{(z^2+1)} dz = \frac{1}{2i}\int_C \frac{f(z)}{z-i} dz - \frac{1}{2i}\int_C \frac{f(z)}{z+i} dz$$

where $f(z) = z/z^2-9$

$f(i) = \frac{i}{-10}$, $f(-i) = \frac{-i}{-8} = \frac{i}{8}$

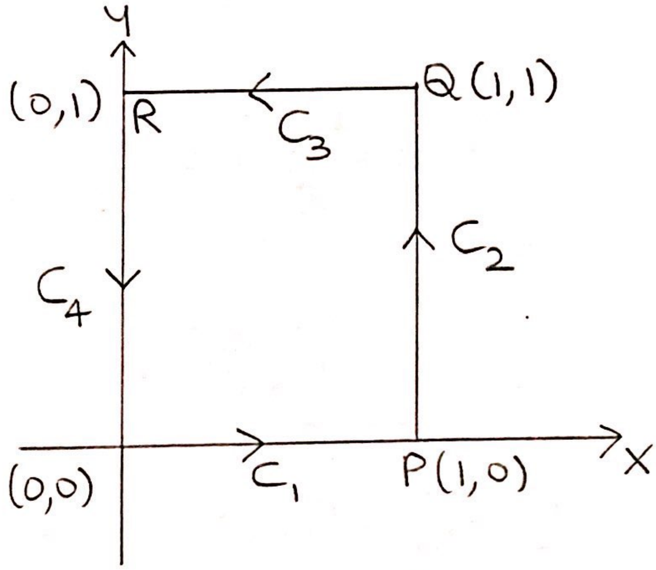
$$\int_C \frac{z}{(z^2-9)(z^2+1)} dz = \frac{1}{2i} \times 2\pi i f(i) - \frac{1}{2i} \times 2\pi i f(-i)$$

Mk

$$= \pi\frac{(i)}{-10} - \pi\frac{(i)}{10} = -\frac{\pi}{5}i$$

4c. Evaluate $\int_C |z|^2\, dz$ where C is a square with the vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Sol^n:



$$\int_C |z|^2\, dz = \int_{C_1} |z|^2\, dz + \int_{C_2} |z|^2\, dz + \int_{C_3} |z|^2\, dz + \int_{C_4} |z|^2\, dz \quad \longrightarrow (1)$$

Along OP (C_1), $y = 0 \Rightarrow dy = 0$

$|z|^2\, dz = x^2\, dx$ where $0 \le x \le 1$

Along PQ (C_2), $x = 1 \Rightarrow dx = 0$

$|z|^2\, dz = (1+y^2)\, i\, dy$ where $0 \le y \le 1$

Along QR (C_3), $y = 1 \Rightarrow dy = 0$

$|z|^2\, dz = (x^2+1)\, dx$, x varies from 1 to 0.

Along RO (C_4), $x = 0 \Rightarrow dx = 0$

$|z|^2\, dz = y^2\, (i\, dy)$, y varies from 1 to 0.

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Using these results in (1), we obtain

$$\int_C |z|^2 dz = \int_{x=0}^{1} x^2 dx + i\int_{y=0}^{1} (1+y^2)\, dy + \int_{x=1}^{0} (x^2+1)\, dx$$
$$+ i\int_{y=1}^{0} y^2 dy$$

$$\int_C |z|^2 dz = \left[\frac{x^3}{3}\right]_0^1 + i\left[y+\frac{y^3}{3}\right]_{y=0}^{1} + \left[\frac{x^3}{3}+x\right]_{x=0}^{1}$$
$$+ i\left[y^3/3\right]_{y=0}^{1}$$

$$= \frac{1}{3}[1^3-0] + i\left[1+\frac{1}{3}-0\right] + \frac{1}{3}[1^3-0] + [1-0]$$
$$+ \frac{i}{3}[1^3-0]$$

$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = -1+i$$

Thus, $\int_C |z|^2 dz = -1+i$ along the given path.

Mrf

MODULE - 3

5a. The probability density function of a variate x given by the following table :

X	-3	-2	-1	0	1	2	3
P(X)	K	2K	3K	4K	3K	2K	K

Find the value of K, mean and variance.

Sol^n: We must have $p(x) \geq 0$ for all x and $\Sigma p(x) = 1$.

$$k + 2k + 3k + 4k + 3k + 2k + k = 1$$

$$16k = 1 \Rightarrow k = 1/16$$

x	-3	-2	-1	0	1	2	3
P(X)	1/16	2/16	3/16	4/16	3/16	2/16	1/16

$$\text{Mean } (\mu) = \Sigma x\, p(x) = \frac{1}{16}(-4-3-3+0+3+4+3)$$

$$\mu = 0$$

$$\text{Variance} = V = \Sigma (x-\mu)^2 . p(x)$$

$$V = \frac{1}{16}(9+8+3+0+3+8+9) = \frac{40}{16} = \underline{\underline{\frac{5}{2}}}$$

5b. After the appointment of a new sales manager, the sales in a two-wheeler showroom is exponentially distributed with mean 4. If two days are selected at random, what is the probability that (i) on both days the sales is over 5 units, (ii) the sales is over 5 units on atleast one of the two days?

Muf

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Sol^n: Mean is $\frac{1}{\alpha} = 4$. The exponential probability function is,

$$f(x) = \alpha e^{-\alpha x}, \quad x > 0$$

$$f(x) = \frac{1}{4} e^{-x/4}$$

(i) The probability that the sales is over 5 units on a randomly chosen days is,

$$p(x>5) = \int_5^\infty \frac{1}{4} e^{-x/4}\, dx$$

$$= e^{-5/4} = 0.2865 = p, \text{ (say)}$$

Accordingly, if two days are chosen, the probability that the sales is over 5 units on both days is

$$p^2 = (e^{-5/4})^2 = e^{-5/2} = 0.082$$

Next, the probability that the sales is over 5 units on atleast one of the two days is, by binomial distribution,

$$b(2,p,1) + b(2,p,2) = {}^2C_1\, p(1-p) + {}^2C_2\, p^2$$

$$= 2p(1-p) + p^2$$

$$= p(2-p) = e^{-5/4}(2 - e^{-5/4})$$

$$= 0.2865 \times 1.7135 = \underline{\underline{0.4909}}$$

Mk

5c. In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D. of 60 hours. If a firm purchases 2500 bulbs, find the number of bulbs that are likely to last for
(i) more than 2100 hours (ii) less than 1950 hours
(iii) between 1900 to 2100 hours.

Solⁿ: By data, $\mu = 2000$, $\sigma = 60$

Standard normal variate, $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-2000}{60}$

(i) To find $P(x > 2100)$

If, $x = 2100$, $z = \dfrac{100}{60} = 1.67$

$P(x > 2100) = P(z > 1.67) = P(z \geq 0) - P(0 < z < 1.67)$

$= 0.5 - \phi(1.67) = 0.5 - 0.4525$

$= 0.0475$

∴ number of bulbs that are likely to last for more than 2100 hours is $2500 \times 0.0475 = 118.75$

$\simeq 119$

(ii) To find $P(x < 1950)$

If, $x = 1950$, $z = -5/6 = -0.83$

$P(x < 1950) = P(z < -0.83) = P(z > 0.83)$

$= P(z \geq 0) - P(0 < z < 0.83)$

$= 0.5 - \phi(0.83) = 0.5 - 0.2967$

$= 0.2033$

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$\therefore$ number of bulbs that are likely to last for more than 1950 hours is 2500×0.2033

$$= 508.25 \simeq 508$$

(iii) To find $P(1900 < x < 2100)$

If $x = 1900$, $z = -1.67$ and if $x = 2100$, $z = 1.67$

$$P(1900 < x < 2100) = P(-1.67 < z < 1.67)$$
$$= 2P(0 < z < 1.67)$$
$$= 2\phi(1.67) = 2 \times 0.4525$$
$$= 0.905$$

$\therefore$ number of bulbs that are likely to last between 1900 and 2100 hours is

$$= 2500 \times 0.905 = 2262.5 \simeq \underline{\underline{2263}}$$

OR

6a. The probability density function of a continuous random variable is

$$f(x) = \begin{cases} kx & , 0 \leq x \leq 5 \\ 0 & , \text{elsewhere} \end{cases}$$

Evaluate 'k' and find (i) $P(1 \leq x \leq 3)$ (ii) $P(x \leq 3)$

Soln: We know that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\therefore \int_0^5 kx\, dx = 1 \Rightarrow k\left[\frac{x^2}{2}\right]_{x=0}^{5} = 1$$

$$\therefore \frac{k}{2}\left[5^2 - 0\right] = 1 \Rightarrow k = \frac{2}{25}$$

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(i) $P(1 \le x \le 3) = \int_1^3 f(x)\,dx = \int_1^3 kx\,dx$

$= k\left[\frac{x^2}{2}\right]_{x=1}^{3} = \frac{2}{25}\left[\frac{1}{2}(3^2-1^2)\right]$

$= \frac{1}{25}[9-1] = \frac{8}{25}$

(ii) $P(x \le 3) = \int_0^3 kx\,dx = k\int_0^3 x\,dx$

$= k\left[\frac{x^2}{2}\right]_{x=0}^{3} = \frac{2}{25} \times \frac{1}{2}\left[3^2-0\right]$

$= \frac{9}{25}$

6b. While forming 36-digit binary numbers a malfunctioning digital computer has found to form a digit incorrectly about 1 times in 1000. Assuming the errors in forming different digits are independent, find the probability of having (i) zero (ii) more than one and (iii) less than five incorrect digits in a given 36-digit number.

Solⁿ: The probability that a digit is incorrectly formed is given as $1/1000 = 0.001$.

Also, $n = 36$.

$\therefore \mu = np = 36 \times 0.001 = 0.036$

The Poisson distribution is, $P(x) = \frac{m^x e^{-m}}{x!}$

MK

$$= \frac{e^{-0.036}(0.036)^x}{x!}$$

(i) the probability having no incorrect digit is,

$$P(x=0) = \frac{e^{-0.036}(0.036)^0}{0!} = e^{-0.036} = \underline{\underline{0.9646}}$$

(ii) $P(x>1) = 1 - P(x \le 1)$

$$= 1 - p(0) - p(1)$$

$$= 1 - 0.9646 - \frac{(0.036)^1}{1!} \times e^{-0.036}$$

$$= 1 - 0.9646 - 0.0347$$

$$= \underline{\underline{0.0007}}$$

(iii) $P(x<5) = P(0) + P(1) + P(2) + P(3) + P(4)$

$$= 0.9646 + 0.0347 + \frac{e^{-0.036}(0.036)^2}{2!}$$

$$+ \frac{e^{-0.036}}{3!}(0.036)^3 + \frac{e^{-0.036}}{4!}(0.036)^4$$

$$= 0.9646 + 0.0347 + 0.00625$$

$$+ 0.000075 + 0.00000068$$

$$= 1.00562568$$

$$\simeq \underline{\underline{1}}$$

Muf

6c. The weekly wages of 1000 workers in an electric company are normally distributed with mean of Rs. 700 and standard deviation of Rs. 50. Estimate the number of workers whose weekly wages will be

(i) between Rs. 700 and Rs. 720,

(ii) more than Rs. 750

Solⁿ: By data, $\mu = 700$ and $\sigma = 50$. Therefore, if x is the weekly wage, the corresponding standard normial variate is,

$$z = \frac{x-\mu}{\sigma} = \frac{x-700}{50}$$

(i) For $x = 700$, $z = 0$ & $x = 720$, $z = \frac{720-700}{50}$

$\therefore z = 0.4$.

$P(700 < x < 720) = P(0 < z < 0.4) = \phi(0.4)$

$= 0.1554$ [By referring normal probability table]

Therefore, among 1000 workers, the estimated number of workers whose weekly wages are between Rs. 700 and Rs. 720 is

$= 0.1554 \times 1000 = 155.4 \simeq 155$

(ii) $P(x > 750) = P(z > 1) = P(z \geq 0) - P(0 < z < 1)$

$= 0.5 - A(1) = 0.5 - 0.3413 = 0.1587$

For 1000 workers, the weekly wages is

$= 0.1587 \times 1000 = 158.7 \simeq \underline{\underline{159}}$

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Module - 4

7a. An experiment on lifetime (t) of cutting tool at different cutting speeds v (units) are given

Speed (v)	350	400	500	600
Life (t)	61	26	7	2.6

Fit a relation of the form $v = at^b$.

Sol^n: Consider, $v = at^b$

$$\log_e v = \log_e a + b \log_e t$$

Let, $V = \log_e v$, $A = \log_e a$, $T = \log_e t$

$$\therefore V = A + bT$$

The normal equations are,

$$\left.\begin{aligned} \Sigma V &= nA + b\Sigma T \\ \Sigma VT &= A\Sigma T + b\Sigma T^2 \end{aligned}\right\} \longrightarrow (I)$$

v	t	$V = \log_e v$	$T = \log_e t$	VT	T^2
350	61	5.8579	4.1109	24.0812	16.8995
400	26	5.9915	3.2581	19.5209	10.6152
500	7	6.2146	1.9459	12.093	3.7865
600	2.6	6.3969	0.9555	6.1122	0.913
		$\Sigma V = 24.4609$	$\Sigma T = 10.2704$	$\Sigma VT = 61.8073$	$\Sigma T^2 = 32.2142$

Eqn (I) becomes, $4A + 10.2704b = 24.4609$

$$10.2704A + 32.2142b = 61.8073$$

on solving, $A = 6.5539$ and $b = -0.1709$

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$A = \log_e a \Rightarrow a = e^A = e^{6.5539} = 701.9766$

Thus, the required curve of fit is,

$$V = (701.9766)\, t^{-0.1709}$$

7b. Fit a best fitting parabola $y = ax^2 + bx + c$ for the following data

x	1	1.5	2	2.5	3	3.5	4
y	1.1	1.3	1.6	2	2.7	3.4	4.1

Soln: Let, $y = ax^2 + bx + c$ be the second degree parabola.

The normal equations are:

$$\Sigma y = a\Sigma x^2 + b\Sigma x + nc$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$$

x	y	xy	x^2y	x^2	x^3	x^4
1	1.1	1.1	1.1	1	1	1
1.5	1.3	1.95	2.925	2.25	3.375	5.0625
2	1.6	3.2	6.4	4	8	16
2.5	2	5	12.5	6.25	15.625	39.0625
3	2.7	8.1	24.3	9	27	81
3.5	3.4	11.9	41.65	12.25	42.875	150.0625
4	4.1	16.4	65.6	16	64	256
$\Sigma x = 17.5$	$\Sigma y = 16.2$	$\Sigma xy = 47.65$	$\Sigma x^2y = 154.4$	$\Sigma x^2 = 50.75$	$\Sigma x^3 = 161.875$	$\Sigma x^4 = 548.1875$

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The normal equations are,

$$50.75a + 17.5b + 7c = 16.2$$

$$161.875a + 50.75b + 17.5c = 47.65$$

$$548.1875a + 161.875b + 50.75c = 154.475$$

on solving,

$$a = 0.2428 \simeq 0.243$$

$$b = -0.1928 \simeq -0.193$$

$$c = 1.0357 \simeq 1.036$$

Thus, the required second degree parabola of fit is,

$$y = 0.243x^2 - 0.193x + 1.036$$

7c. Ten competitors in music contest are ranked by 3 judges A, B, C in the following order. Use the rank correlation coefficient to decide which pair of judges have the most approach (nearest approach) to common taste of music.

A	1	6	5	10	3	2	4	9	7	8
B	3	5	8	4	7	10	2	1	6	9
C	6	4	9	8	1	2	3	10	5	7

Sol^n: We have,

$$\rho = 1 - \frac{6\Sigma d^2}{n(n^2-1)} \quad \text{and} \quad n = 10$$

We compute ρ_{AB}, ρ_{BC} & ρ_{CA} with the help of the following table.

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A	B	C	d_{AB}^2	d_{BC}^2	d_{CA}^2
1	3	6	4	9	25
6	5	4	1	1	4
5	8	9	9	1	16
10	4	8	36	16	4
3	7	1	16	36	4
2	10	2	64	64	0
4	2	3	4	1	1
9	1	10	64	81	1
7	6	5	1	1	4
8	9	7	1	4	1
			$\Sigma d_{AB}^2 = 200$	$\Sigma d_{BC}^2 = 214$	$\Sigma d_{CA}^2 = 60$

$$\rho_{AB} = 1 - \frac{6(200)}{10(10^2-1)} = -0.21$$

$$\rho_{BC} = 1 - \frac{6(214)}{10(10^2-1)} = -0.297$$

$$\rho_{CA} = 1 - \frac{6(60)}{10(10^2-1)} = 0.636$$

Since, ρ_{AB} and ρ_{BC} are negative, it means that their tastes are opposite. But, ρ_{CA} is positive & is nearer to 1. Thus, we conclude that the judges C and A have the nearest approach to common taste of music.

8a. $8x-10y+66=0$, $40x-18y=214$ are the two regression lines. Find the mean of x's, y's and the correlation coefficient. Find σ_y if $\sigma_x=3$.

Solⁿ: We know that regression lines passes through $\bar{x}$ and $\bar{y}$.

$$8\bar{x}-10\bar{y}=-66$$
$$40\bar{x}-18\bar{y}=214$$

Solving, we get $\bar{x}=13$, $\bar{y}=17$

Rewriting the given equations to find the regression coefficients.

$10y=8x+66$ or $y=0.8x+66 \longrightarrow (1)$

$40x=18y+214$ or $x=0.45y+5.35 \longrightarrow (2)$

From (1) & (2), $r\frac{\sigma_y}{\sigma_x}=0.8$; $r\frac{\sigma_x}{\sigma_y}=0.45$

Correlation coefficient $r=\sqrt{0.8\times0.45}=\pm0.6$

Thus, $r=0.6$ since both the regression coefficients are positive.

Also, $\sigma_x=3$ by data and $r\frac{\sigma_y}{\sigma_x}=0.8$ gives

$$0.6\,\sigma_y=2.4.$$

Thus, $\sigma_y=4$

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8b. Find the equation of the best fitting straight line for the following data and hence estimate the value of the dependent variable corresponding to the value 30 of the independent variable.

x	5	10	15	20	25
y	16	19	23	26	30

Solⁿ: Let, $y = ax + b$ be the equation of the best fitting straight line.

The normal equations are,

$$\Sigma y = a\Sigma x + nb$$

$$\Sigma xy = a\Sigma x^2 + b\Sigma x$$

x	y	xy	x^2
5	16	80	25
10	19	190	100
15	23	345	225
20	26	520	400
25	30	750	625
$\Sigma x = 75$	$\Sigma y = 114$	$\Sigma xy = 1885$	$\Sigma x^2 = 1375$

$$75a + 5b = 114$$

$$1375a + 75b = 1885$$

on solving, $a = 0.7$, $b = 12.3$

$$\underline{\underline{y = 0.7x + 12.3}}$$

When $x = 30$, we obtain $y = 0.7(30) + 12.3 = \underline{\underline{33.3}}$

$$\therefore \underline{\underline{y = 33.3}}$$

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8c. Obtain the lines of regression and hence find the coefficient of correlation for the data.

x	1	3	4	2	5	8	9	10	13	15
y	8	6	10	8	12	16	16	10	32	32

Soln: $\bar{x} = \frac{\Sigma x}{10} = \frac{70}{10} = 7$, $\bar{y} = \frac{\Sigma y}{n} = \frac{150}{10} = 15$

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	Y^2	XY
1	8	-6	-7	36	49	42
3	6	-4	-9	16	81	36
4	10	-3	-5	9	25	15
2	8	-5	-7	25	49	35
5	12	-2	-3	4	9	6
8	16	1	1	1	1	1
9	16	2	1	4	1	2
10	10	3	-5	9	25	-15
13	32	6	17	36	289	102
15	32	8	17	64	289	136
$\Sigma x = 70$	$\Sigma y = 150$			$\Sigma X^2 = 204$	$\Sigma Y^2 = 818$	$\Sigma XY = 360$

We have lines of regression in the form

$$Y = \frac{\Sigma XY}{\Sigma X^2} \times X \quad ; \quad X = \frac{\Sigma XY}{Y^2} \cdot Y$$

$$y - 15 = \frac{360}{204}(x-7) \quad ; \quad x - 7 = \frac{360}{818}(y-15)$$

$$y - 15 = 1.76(x-7) \quad ; \quad x - 7 = 0.44(y-15)$$

$$y = 1.76x + 2.68 \quad ; \quad x = 0.44y + 0.4$$

Mg

$$\therefore r = \sqrt{(\text{Coeff. of } x)(\text{coeff. of } y)}$$

$$= \sqrt{(1.76)(0.44)} = +0.88$$

The sign of 'r' is positive since both the regression coefficients are positive.

Thus, $\underline{\underline{r = 0.88}}$

MODULE-5

9a. The manufacturer of a certain medicine claimed that it was 90% effective in relieving an allergy for a period of 8 hours. In a sample of 200 persons who had the allergy, the medicine provided relief for 160 persons. Determine whether the manufacturer's claim is legitimate at 0.01 level of significance.

Sol^n: Let 'p' denote the probability that a person gets relief from the allergy by using the medicine, and X denote the number of persons in the sample of $N = 200$ persons who got relief by using the medicine.

$$p = 0.9, \quad \mu = Np = 200 \times 0.9 = 180$$

$$\sigma_x = \sqrt{Npq} = \sqrt{200 \times 0.9 \times 0.1} = 4.24$$

Given, $X = 160$. $\quad z = \dfrac{X - \mu}{\sigma_x} = \dfrac{160 - 180}{4.24} = -4.72$

Mey

$z = -4.72$ lies outside $(-2.58, 2.58)$ at 0.01 level of significance. Therefore, on the basis of the two-tailed test, the hypothesis is rejected at 0.01 level of significance. Accordingly, the manufacturer's claim is not legitimate at 0.01 level of significance.

9b. A joint probability distribution is given by the following table.

X \ Y	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Find (a) Marginal distributions of X and Y
(b) COV(X,Y) (c) Correlation of X and Y

Sol^n: Marginal distributions of X and Y are

x_i	1	3
$f(x_i)$	0.5	0.5

y_j	-3	2	4
$g(y_j)$	0.4	0.3	0.3

(b) $\mu_X = E(X) = \Sigma x_i f(x_i)$

$= (1) \times (0.5) + 3(0.5) = 2$

$\mu_Y = E(Y) = \Sigma y_j g(y_j)$

$= (-3)(0.4) + 2(0.3) + 4(0.3)$

$= 0.6$

Mey

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$$E(XY) = \sum x_i y_j J(y_j)$$

$$= 1\times(-3)\times(0.1) + 1\times 2\times(0.2)$$

$$+ 1\times 4\times 0.2 + 3(-3)(0.3)$$

$$+ (3)(2)(0.1) + 3\times 4\times 0.1$$

$$= 0$$

$$\text{cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$= 0 - 2\times 0.6 = \underline{\underline{-1.2}}$$

(c) $\sigma_X^2 = E(X^2) - \mu_X^2$ and $\sigma_Y^2 = E(Y^2) - \mu_Y^2$

$$E(X^2) = \sum x_i^2 f(x_i)$$

$$= 1\times 0.5 + (3)^2\times 0.5 = 5$$

$$E(Y^2) = \sum y_j^2 g(y_j)$$

$$= (-3)^2\times 0.4 + (4\times 0.3) + (16\times 0.3)$$

$$= 9.6$$

$\therefore \sigma_X^2 = 5 - 2^2 = 5 - 4 = 1$ $\boxed{\therefore \sigma_X = 1}$

$\sigma_Y^2 = 9.6 - 0.6 = 9$ $\boxed{\therefore \sigma_Y = 3}$

Correlation of X and Y $= \rho(X, Y) = \dfrac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$

$$= \frac{-1.2}{1\times 3} = \underline{-0.4}$$

Mwf

Dr. Meenal Kaliwal

9c. It has been found from experience that the mean breaking strength of a particular brand of thread is 275.6 gms with standard deviation of 39.7 gms. Recently a sample of 36 pieces of thread showed a mean breaking strength of 253.2 gms. Can one conclude at a significance level of (a) 0.05 (b) 0.01 that the thread has become inferior?

Soln: $H_0 : \mu = 275.6$ gms, mean breaking strength

$H_1 : \mu < 275.6$ gms, inferior in breaking strength

Mean breaking strength of a sample of 36 pieces $= 253.2$

$\therefore$ difference $= \underline{275.6 - 253.2} = 22.4$

$n = 36$

$$Z = \frac{\text{Difference}}{\sigma/\sqrt{n}} = \frac{22.4}{39.7/6} = \underline{\underline{3.38}}$$

The value of Z is greater than the critical value of $Z = 1.645$ at 5% level and 2.33 at 1% level of significance.

H_1 is accepted at both 0.05 and 0.01 levels in accordance with one tailed test.

The thread has become inferior.

Mey

10a. A coin is tossed 1000 times and head turns up 540 times. Decide on the hypothesis that the coin is unbiased.

Solⁿ: Let us suppose that the coin is unbiased.

p = probability of getting a head in one toss

$= 1/2$.

Since, $p+q=1$, $q=1/2$

Expected number of heads in 1000 toses

$= np = 1000 \times \frac{1}{2} = 500$

Actual number of heads $= 540$

The difference $= x - np = 540 - 500 = 40$.

$$z = \frac{x-np}{\sqrt{npq}} = \frac{40}{\sqrt{1000 \times \frac{1}{2} \times \frac{1}{2}}} = 2.53 < 2.58$$

Thus, we can say that the coin is unbiased.

10b. A sample of 100 tyres is taken from a lot. The mean life of tyres is found to be 39350, Kilometres with a standard deviation of 3260. Can it be considered as a true random sample from a population withe mean life of 40,000 kilometres. Use 0.05 level of significance. Establish 99% confidence limits within which the mean life of tyres expected to lie.

(Given that $z_{0.05} = 1.96$, $z_{0.01} = 2.58$)

Mk

Sol^n: By data $n = 100$ and $\bar{x} = 39350$ kms

Null hypothesis : $H_0 : \mu = 40{,}000$ kms and

$\sigma = 3260$ kms

that is the claim that the sample has been drawn from a population of tyres which has a mean life of 40,000 kms and a standard deviation of 3260 km is true.

Alternate hypothesis ; $H_1 : \mu \neq 40{,}000$ kms

(two-tailed test)

Level of significance : $\alpha = 0.05$

Test stastic : Since n is large, under H_0 the test

$$|z| = \left| \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right| = \left| \frac{39350 - 40{,}000}{\frac{3260}{\sqrt{100}}} \right|.$$

$$= \left| \frac{-650}{\frac{3260}{10}} \right| = \left| \frac{-650}{3260} \times 10 \right| = |-1.9938|$$

$$= 1.9938 > 1.96$$

The value of 'z' is significant at 5% level of significant, therefore null hypothesis is rejected.

We may conclude that the claim that the sample has been drawn from a population of tyres which has a mean life of 40,000 kms & standard deviation of 3260 km is not valid.

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To find 99% confidence limits, it is given by

$$\bar{x} \pm 2.58 \frac{\sigma}{\sqrt{n}} = 39350 \pm 2.58\,(326)$$

$$= 39350 + 841,\ 39350 - 841$$

$$= \underline{\underline{40{,}191,\ 38509}}$$

10 c. Fit a Poisson's distribution to the following data and test the goodness of fit at 5% level of significance, given that $\chi^2_{(0.05)}(3) = 7.815$ for 3 degrees of freedom.

x	0	1	2	3	4
frequency	122	60	15	2	1

Soln: $\Sigma f_i = 200$; $\Sigma f_i x_i = 100$

$$\text{Mean} = m = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{100}{200} = \frac{1}{2} = 0.5$$

$$\therefore e^{-m} = e^{-0.5} = 0.6065$$

Probability density function is $\frac{e^{-m} m^x}{x!}$

The corresponding theoretical frequencies is,

$$F_i = \Sigma f_i \frac{e^{-m} m^x}{x!} = 200 \frac{e^{-m} m^x}{x!}$$

i.e. $$F_i = \frac{200\,(0.6065)\,(0.5)^x}{x!}$$

MK

x	0	1	2	3	4
F_i	121	61	15.16	2.53	0.32

We note that the sum of the theoretical frequencies are,

$$\Sigma F_i = 121 + 61 + 15.16 + 2.53 + 0.32$$

$$= 200.01$$

We observe that $\Sigma f_i \neq \Sigma F_i$ with a difference of 0.01 so we modify the last theoretical frequency as $F_5 = 0.32 - 0.01 = 0.31$

We know that, $\chi^2 = \dfrac{\Sigma (O_i - E_i)^2}{E_i}$

$$\chi^2 = \frac{(122-121)^2}{121} + \frac{(60-61)^2}{61} + \frac{(15-15.16)^2}{15.16} + \frac{(2-2.53)^2}{2.53} + \frac{(1-0.32)^2}{0.32}$$

$$= \frac{1}{121} + \frac{1}{61} + \frac{0.0256}{15.16} + \frac{0.2809}{2.53} + \frac{0.4624}{0.32}$$

$$= 0.0082 + 0.0163 + 0.0016 + 0.1110 + 1.445$$

$$\underline{\underline{\chi^2}} = 1.5821$$

Degree of freedom $= \nu = n - k = 5 - 2 = 3$

For, $\nu = 3 \quad \chi^2_{0.05} = 7.815$

Since, the calculated value $1.5821 < 7.815$ the tabulated value, the Poisson's distribution is a good fit.

My

← END →