CBCS SCHEME

USN | | | | | | | | | | | **18MAT31**

Third Semester B.E. Degree Examination, Jan./Feb. 2021

Transform Calculus, Fourier Series and Numerical Techniques

Time: 3 hrs. **Max. Marks: 100**

Note: ***Answer any FIVE full questions, choosing ONE full question from each module.***

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

Module-1

1 a. Find the Laplace transform of $\cos t \ \cos 2t \ \cos 3t$. **(06 Marks)**

b. If $f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases}$ and $f(t + 2a) = f(t)$, show that $L\{f(t)\} = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$. **(07 Marks)**

c. Find the Inverse Laplace transforms of :

i) $\dfrac{2s+1}{s^2+6s+13}$ ii) $\dfrac{1}{3}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$. **(07 Marks)**

OR

2 a. Express the function f(t) in terms of unit step function and find its Laplace transform, where

$$f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ t^2, & t > 2 \end{cases}$$ **(06 Marks)**

b. Find the Inverse Laplace transform of $\dfrac{s^2}{(s^2+a^2)^2}$ using Convolution theorem. **(07 Marks)**

c. Solve by the method of Laplace transforms, the equation
$y'' + 4y' + 3y = e^{-t}$ given $y(0) = 0$, $y'(0) = 0$. **(07 Marks)**

Module-2

3 a. Obtain the Fourier series of the function $f(x) = x^2$ in $-\pi \le x \le \pi$. **(06 Marks)**

b. Obtain the Fourier series expansion of

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ x - 2\pi, & \pi < x < 2\pi \end{cases}$$ **(07 Marks)**

c. Find the Cosine half range series for $f(x) = x(\ell - x)$, $0 \le x \le \ell$. **(07 Marks)**

OR

4 a. Obtain the Fourier series of $f(x) = |x|$ in $(-\ell, \ell)$. **(06 Marks)**

b. Find the sine half range series for

$$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$$ **(07 Marks)**

c. Obtain the constant term and the coefficients of the first cosine and sine terms in the Fourier expansion of y from the table. **(07 Marks)**

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Module-3

5 a. If $f(x) = \begin{cases} 1 - x^2 & , |x| < 1 \\ 0 & , |x| \geq 1 \end{cases}$. Find the Fourier transform of f(x) and hence find value of $\int_0^\infty \frac{x\cos x - \sin x}{x^3} dx$. (06 Marks)

b. Find the Fourier Cosine transform of

$$f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4 - x, & 1 < x < 4 \\ 0, & x > 4 \end{cases}.$$ (07 Marks)

c. Find the Z – transform of $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$. (07 Marks)

OR

6 a. Solve the Integral equation

$$\int_0^\infty f(\theta)\cos\alpha\theta \, d\theta = \begin{cases} 1 - \alpha & , 0 \leq \alpha \leq 1 \\ 0 & , \alpha > 1 \end{cases}$$ hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} dt$. (06 Marks)

b. Find the Inverse Z – transform of $\frac{2z^2 + 3z}{(z + 2)(z - 4)}$. (07 Marks)

c. Using the Z – transform, solve $Y_{n+2} - 4Y_n = 0$, given $Y_0 = 0$, $Y_1 = 2$. (07 Marks)

Module-4

7 a. Using Taylor's series method, solve the Initial value problem $\frac{dy}{dx} = x^2 y - 1$, $y(0) = 1$ at the point $x = 0.1$. Consider upto 4^{th} degree term. (06 Marks)

b. Use modified Euler's method to compute y(0.1), given that $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ by taking h = 0.05. Consider two approximations in each step (07 Marks)

c. Given that $\frac{dy}{dx} = x - y^2$, find y at x = 0.8 with

x :	0	0.2	0.4	0.6
y :	0	0.02	0.0795	0.1762

By applying Milne's method. Apply corrector formula once. (07 Marks)

OR

8 a. Solve the following by Modified Euler's method $\frac{dy}{dx} = x + |\sqrt{y}|$, y(0) = 1 at x = 0.4 by taking h = 0.2. Consider two modifications in each step. (06 Marks)

b. Given $\frac{dy}{dx} = 3x + \frac{y}{2}$, y(0) = 1. Compute y(0.2) by taking h = 0.2 using Runge – Kutta method of order IV. (07 Marks)

c. Given $\frac{dy}{dx} = (1 + y)x^2$ and y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979, determine y(1.4) by Adam's Bashforth method. Apply corrector formula once. (07 Marks)

Module-5

9 a. Given $y'' - xy' - y = 0$ with $y(0) = 1$, $y'(0) = 0$. Compute $y(0.2)$ using Runge – Kutta method. (06 Marks)

b. Derive Euler's equation in the form $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$. (07 Marks)

c. Prove that the geodesics on a plane are straight lines. (07 Marks)

OR

10 a. Find the curve on which functional

$\int_0^1 [(y')^2 + 12xy]dx$ with $y(0) = 0$, $y(1) = 1$ can be extremized. (06 Marks)

b. Obtain the solution of the equation $\frac{2d^2y}{dx^2} = 4x + \frac{dy}{dx}$ by computing the value of dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the following data. Apply corrector formula once. (07 Marks)

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

c. A heavy cable hangs freely under gravity between two fixed points. Show that the shape of the cable is Catenary $y = c\cosh\left(\frac{x+a}{c}\right)$ (07 Marks)

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Third Semester B.E. Degree Examination (Jan/Feb 2021)

Transform Calculus, Fourier Series & Numerical Techniques

— SCHEME AND SOLUTION —

1 a) Let $f(t) = \cos t \cdot \cos 2t \cdot \cos 3t$

$$= \frac{1}{2}\cos t[\cos 5t + \cos t] \text{ — (1m)}$$

$$= \frac{1}{2}[\cos t \cdot \cos 5t + \cos^2 t]$$

$$= \frac{1}{2}\left[\frac{1}{2}\{\cos 6t + \cos 4t\} + \frac{1}{2}\{1 + \cos 2t\}\right] \text{ — (1m)}$$

$$f(t) = \frac{1}{4}[1 + \cos 2t + \cos 4t + \cos 6t] \text{ — (1m)}$$

$$\therefore L[f(t)] = \frac{1}{4}[L(1) + L(\cos 2t) + L[\cos 4t] + L(\cos 6t)] \text{ — (1m)}$$

$$\text{ie } L[f(t)] = \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+4} + \frac{s}{s^2+16} + \frac{s}{s^2+36}\right] \text{ — (2m)}$$

6M

b) $$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt \;;\; T = 2a$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt \text{ — (1m)}$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a t e^{-st}\,dt + \int_a^{2a}(2a-t)\,e^{-st} f(t)\,dt\right] \text{ — (1m)}$$

Applying Bernoulli's rule.

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left[\left\{t\cdot\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right\}_0^a + \left\{(2a-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right\}_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-1}{s}(ae^{-as} - 0) - \frac{1}{s^2}(e^{-as} - 1) - \frac{1}{s}(0 - ae^{-as}) + \frac{1}{s^2}(e^{-2as} - e^{-as})\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}(1 - 2e^{-as} + e^{-2as}) = \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})} \text{ — (3m)}$$

$$= \frac{(1-e^{-as})}{s^2(1+e^{-as})} = \frac{e^{as/2} - e^{-as/2}}{s^2(e^{as/2} + e^{-as/2})} = \frac{2\sinh(as/2)}{s^2 \cdot 2\cosh(as/2)}$$

$$\text{Thus } L[f(t)] = \frac{1}{s^2}\tanh(as/2) \text{ — (2m)}$$

07M

1 c) i) $\frac{2s+1}{s^2+6s+13} = \frac{2s+1}{(s+3)^2+4} = \frac{2(s+3)-5}{(s+3)^2+4}$

$= \frac{2(s+3)}{(s+3)^2+4} - \frac{5}{(s+3)^2+4}$ — (1m)

$L^{-1}\left[\frac{2s+1}{s^2+6s+13}\right] = 2L^{-1}\left[\frac{(s+3)}{(s+3)^2+4}\right] - 5L^{-1}\left[\frac{1}{(s+3)^2+4}\right]$ — (1m)

$= 2e^{-3t}L^{-1}\left[\frac{s}{s^2+4}\right] - 5e^{-3t}L^{-1}\left[\frac{1}{s^2+4}\right]$

$= 2e^{-3t}\cdot\cos 2t - \frac{5}{2}e^{-3t}\sin 2t$ — (1m)

ii) $\bar{f}(s) = \frac{1}{3}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) = \frac{1}{3}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]$

$\bar{f}'(s) = \frac{1}{3}\left[\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right] = \frac{2}{3}\left[\frac{s}{s^2+b^2} - \frac{s}{s^2+a^2}\right]$

$-\bar{f}'(s) = \frac{2}{3}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$ — (2m)

$L^{-1}[-\bar{f}'(s)] = \frac{2}{3}\left[L^{-1}\left\{\frac{s}{s^2+a^2}\right\} - L^{-1}\left\{\frac{s}{s^2+b^2}\right\}\right]$

$tf(t) = \frac{2}{3}[\cos at - \cos bt]$ — (1m)

$f(t) = \frac{2}{3}\left[\frac{\cos at - \cos bt}{t}\right]$ — (1m)

7M

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2 a) By the def^n of unit step function.

$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}u(t-a) + \{f_3(t) - f_2(t)\}u(t-b)$ — (1m)

$= 1 + (t-1)u(t-1) + (t^2-t)u(t-2)$ — (1m)

$\therefore L[f(t)] = L[1] + L[(t-1)u(t-1)] + L[(t^2-t)u(t-2)] \longrightarrow (1)$

Let $F(t-1) = t-1$; $G(t-2) = t^2-t$

$F(t) = t$; $G(t) = (t+2)^2 - (t+2) = t^2+3t+2$ — (2m)

$\bar{F}(s) = \frac{1}{s^2}$; $\bar{G}(s) = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}$

From eq^n (1)

$L[f(t)] = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left\{\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right\}$ — (2m)

6M

2

b) Let $\bar{f}(s) = \bar{g}(s) = \frac{s}{s^2+a^2}$

$\Rightarrow f(t) = L^{-1}[\bar{f}(s)] = \cos at = g(t)$ —— (1m)

By applying Convolution theorem

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \cos au \cdot \cos(at-au)\,du \text{ —— (1m)}$$

$$= \frac{1}{2}\int_{u=0}^{t} [\cos(au+at-au) + \cos(au-at+au)]\,du.$$

$$= \frac{1}{2}\int_{u=0}^{t} [\cos at + \cos(2au-at)]\,du \text{ —— (2m)}$$

$$= \frac{1}{2}\left[\cos at\,(u)_0^t + \left\{\frac{\sin(2au-at)}{2a}\right\}_0^t\right]$$

$$= \frac{1}{2}\left[\cos at\,(t) + \frac{1}{2a}(\sin at + \sin at)\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right] \text{ —— (2m)}$$

Thus, $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}[at\cos at + \sin at]$ —— (1m)

7M

c) Taking Laplace transform on both sides

$$L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = L[e^{-t}]$$

$$[s^2\bar{y}(s) - sy(0) - y'(0)] + 4[\bar{y}(s)s - y(0)] + 3\bar{y}(s) = \frac{1}{s+1} \text{ —— (1m)}$$

$$(s^2+4s+3)\bar{y}(s) = \frac{1}{s+1} + s+1+4 = s+5+\frac{1}{s+1}$$

$$\bar{y}(s) = \frac{s+5}{(s+1)(s+3)} + \frac{1}{(s+1)^2(s+3)} = \frac{s^2+6s+6}{(s+1)^2(s+3)} \text{ —— (1m)}$$

$$\therefore y(t) = L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right] \longrightarrow (1)$$

Let $\frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+3)}$ —— (1m)

$$s^2+6s+6 = A(s+1)(s+3) + B(s+3) + C(s+1)^2$$

put $s=-1$ $\boxed{B = \frac{1}{2}}$

put $s=-3$ $\boxed{C = -3/4}$ —— (2m)

Equating the coefficient of s^2

$A+C=1 \quad \therefore \boxed{A = 7/4}$

$$L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right] = \frac{7}{4}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right]-\frac{3}{4}L^{-1}\left[\frac{1}{s+3}\right]$$

$$y(t)= \frac{7}{4}e^{-t}+\frac{1}{2}e^{-t}t-\frac{3}{4}e^{-3t} \longrightarrow (2m)$$

7M

3 a The Fourier series of $f(x)$ having period 2π is given by,

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n\sin nx \longrightarrow ①$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3\pi} = \frac{2}{3}\pi^2$$

$$\boxed{\frac{a_0}{2} = \frac{\pi^2}{3}} \longrightarrow (1m)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx \quad \text{Integrating by parts.}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi n^2}\left[x\cos nx\right]_{-\pi}^{\pi} = \frac{2}{\pi n^2}\left[\pi\cos n\pi-(-\pi)\cos n\pi\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}} \longrightarrow (2m)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right)-(2x)\left(\frac{-\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}(\pi^2\cos n\pi-\pi^2\cos n\pi)+0+\frac{2}{n^3}(\cos n\pi-\cos n\pi)\right]$$

$$\therefore \boxed{b_n = 0} \longrightarrow (2m)$$

$\therefore$ From eq^n ①

$$f(x) = \frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx \longrightarrow (1m)$$

is the required Fourier series Expansion

6M

3 b) The Fourier series of $f(x)$ having period 2π is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1) \text{ — (1m)}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left\{2\pi x - \frac{x^2}{2}\right\}_{\pi}^{2\pi}\right]$$

$$a_0 = \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - 0\right) + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi$$

$$\boxed{\frac{a_0}{2} = \frac{\pi}{2}} \text{ — (2M)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi} + \left\{(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$\therefore a_n = \frac{1}{\pi n^2}[\cos n\pi - 1] - \frac{1}{\pi n^2}[1 - \cos n\pi]$$

$$= \frac{1}{\pi n^2}(-2 + 2\cos n\pi) = \frac{-2}{\pi n^2}[1 - (-1)^n] \text{ — (2m)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi} + \left\{(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{-1}{\pi n}\left[(x\cos nx)_0^{\pi} + [(2\pi - x)\cos nx]_{\pi}^{2\pi}\right]$$

$$= \frac{-1}{\pi n}[\pi\cos n\pi + (-\pi\cos n\pi)] = 0 \Rightarrow \boxed{b_n = 0} \text{ — (1m)}$$

$\therefore$ From eqn (1)

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2}{\pi n^2}\{1 - (-1)^n\}\cos nx \text{ — (1m)}$$

$\boxed{7M}$

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c) The cosine half range Fourier series of $f(x)$ in $(0, l)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \cancel{dx} \longrightarrow (1)$$

Where $a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$; $a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$. — (1M)

$$a_0 = \frac{2}{l}\int_0^l (lx - x^2)\,dx = \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right] = \frac{l^2}{3} \Rightarrow \boxed{\frac{a_0}{2} = l^2/6} \longleftarrow \text{1m}$$

$$a_n = \frac{2}{l}\int_0^l \underset{①}{(lx - x^2)}\,\underset{②}{\cos\left(\frac{n\pi x}{l}\right)}\,dx \longleftarrow \text{1m}$$

$$= \frac{2}{l}\left[(lx - x^2)\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi x}{l}\right)} - (l - 2x)\frac{-\cos(n\pi x/l)}{(n\pi x/l)^2} + (-2)\frac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^3}\right]_0^l$$

$$= \frac{2}{l}\cdot\frac{l^2}{n^2\pi^2}\left[(l - 2x)\cos(n\pi x/l)\right]_0^l$$

$$= \frac{2l}{n^2\pi^2}(-l\cos n\pi - l) = \frac{-2l^2}{n^2\pi^2}[1 + \cos n\pi] = \frac{-2l^2}{n^2\pi^2}[1 + (-1)^n] \text{ — 2m}$$

Thus the required cosine half range fourier series is

$$f(x) = \frac{l^2}{6} - \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\{1 + (-1)^n\}\cos\left(\frac{n\pi x}{l}\right) \text{ — 2m}$$

7M

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4 a) The Fourier series of $f(x)$ having period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right) \longrightarrow ① \text{ — 1m}$$

$f(x) = |x|$ is an even function $\Rightarrow \boxed{b_n = 0}$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = l$$

ie $\boxed{a_0 = l}$ — 2m

$$a_n = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{\cos(n\pi x/l)}{(n\pi/l)^2}\right]_0^l = \frac{2l^2}{l(n\pi)^2}\left[\cos(n\pi x/l)\right]_0^l$$

$$= \frac{2l}{n^2\pi^2}\left[\cos n\pi - \cos 0\right] = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{-2l}{n^2\pi^2}\left[1 - (-1)^n\right] \text{ —— 2m}$$

Substituting for a_0, a_n, b_n in Eqn (B)

$$f(x) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{-2l}{n^2\pi^2}\{1-(-1)^n\}$$ — (1m)

$\boxed{6M}$

4 b) The sine half range Fourier Series is given by.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$\text{i.e. } b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi-x)\sin nx\,dx\right]$$ — (2m)

$$= \frac{2}{\pi}\left[\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left\{(\pi-x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left\{\frac{-1}{n}(x\cos nx)_0^{\pi/2} + \frac{1}{n^2}[\sin nx]_0^{\pi/2} - \frac{1}{n}[(\pi-x)\cos nx]_{\pi/2}^{\pi} - \frac{1}{n^2}[\sin nx]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi}{2n}\cos(n\pi/2) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) + \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin(n\pi/2)\right\}$$

$$= \frac{2}{\pi}\cdot\frac{2}{n^2}\sin\left(\frac{n\pi}{2}\right) = \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$ — (4m)

$\therefore$ The required sine half range series is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\cdot\sin nx$$ — (1m)

$\boxed{7M}$

c) The values at 0, 1, 2, 3, 4, 5 are given $(N=6) \Rightarrow 0 \le x \le 6$.
$2l = 6 \Rightarrow \boxed{l=3}$. The F.S. of period $2l$ is

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$$

$$\text{i.e. } y = \frac{a_0}{2} + a_1\cos\left(\frac{\pi x}{3}\right) + b_1\sin(\pi x/3) \longrightarrow (1)$$ — (1m)

x	$\theta = \pi x/3$	y	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	0	9	1	9	0	0
1	60	18	0.5	9	0.866	15.588
2	120	24	−0.5	−12	0.866	20.784
3	180	28	−1	−28	0	0
4	240	26	−0.5	−13	−0.866	−22.516
5	300	20	0.5	10	−0.866	−17.32
Total		125		−25		−3.464

(3m)

$$a_0 = \frac{2}{N}\Sigma y = \frac{1}{3}(125) = 41.67 \Rightarrow \frac{a_0}{2} = 20.83$$

$$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{1}{3}(-25) = -8.333$$

$$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{1}{3}(-3.464) = -1.155$$

(2m)

$$\therefore y = 20.83 + (-8.333)\cos\left(\frac{\pi x}{3}\right) + (-1.155)\sin\left(\frac{\pi x}{3}\right)$$ (1m)

7M

5 a)

$$F(u) = \int_{-\infty}^{\infty} f(x)e^{iux}\,dx = \int_{-1}^{1}(1-x^2)e^{iux}\,dx$$

$$= \left[(1-x^2)\frac{e^{iux}}{iu} - (-2x)\frac{e^{iux}}{(iu)^2} + (-2)\frac{e^{iux}}{i^3u^3}\right]_{-1}^{1}$$ (1m)

$$= -\frac{i}{u}\left[(1-x^2)e^{iux}\right]_{x=-1}^{1} - \frac{2}{u^2}\left[xe^{iux}\right]_{-1}^{1} - \frac{2i}{u^3}\left[e^{iux}\right]_{-1}^{1}$$

$$= \frac{-2}{u^2}(e^{iu}+e^{-iu}) - \frac{2i}{u^3}(e^{iu}-e^{-iu})$$

$$\therefore F(u) = \frac{-4\cos u}{u^2} + \frac{4\sin u}{u^3} = 4\left(\frac{\sin u - u\cos u}{u^3}\right)$$ (2M)

By the inverse Fourier transforms formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-iux}\,du$$

If $x=0 \Rightarrow f(0) = 1$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin u - u\cos u}{u^3}\right)du = 1$$

$$\int_{-\infty}^{\infty}\frac{\sin u - u\cos u}{u^3}\,du = \frac{2\pi}{4} = \frac{\pi}{2}$$ (2m)

$$\therefore 2\int_{0}^{\infty}\frac{\sin u - u\cos u}{u^3}\,du = \frac{\pi}{2}$$

$$\therefore \int_{0}^{\infty}\frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}$$

$$\text{ie } \int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\,dx = \frac{-\pi}{4}$$ (2m)

6M

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5 b) $F_c(u) = \int_0^\infty f(x) \cos ux \, dx$

$= \int_0^1 f(x) \cos ux \, dx + \int_1^4 f(x) \cos ux \, dx + \int_4^\infty f(x) \cos ux \, dx$ — 1m

$= \int_0^1 4x \cos ux \, dx + \int_1^4 (4-x) \cos ux \, dx + \int_4^\infty 0 \cdot \cos ux \, dx$ — 2m

i.e. $F_c(u) = \left[4x \cdot \frac{\sin ux}{u} - (4)\left(-\frac{\cos ux}{u^2}\right) \right]_0^1 + \left[(4-x) \frac{\sin ux}{u} - (-1)\left(-\frac{\cos ux}{u^2}\right) \right]_1^4$

$= \frac{4}{u} [x \sin ux]_0^1 + \frac{4}{u^2} [\cos ux]_0^1 + \frac{1}{u} [(4-x) \sin ux]_1^4 - \frac{1}{u^2} [\cos ux]_1^4$ — 1m

$= \frac{4}{u} \sin u + \frac{4}{u^2} \cos u - \frac{4}{u^2} - \frac{3}{u} \sin u - \frac{1}{u^2} \cos 4u + \frac{1}{u^2} \cos u$ — 1m

i.e. $F_c(u) = \frac{1}{u} \sin u + \frac{5\cos u - 4}{u^2} - \frac{1}{u^2} \cos 4u$ — 2m

7M

c) Let $U_n = \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

$= \cos\left(\frac{n\pi}{2}\right) \cos(\pi/4) - \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}} \left[\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right) \right]$ — 1m

$Z[U_n] = \frac{1}{\sqrt{2}} \left[Z\left\{\cos\left(\frac{n\pi}{2}\right)\right\} - Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\} \right] \longrightarrow (1)$

Consider $Z[e^{in\pi/2}] = \frac{z}{z - e^{i\pi/2}} = \frac{z}{z - \cos(\pi/2) - i\sin(\pi/2)} = \frac{z}{z-i}$

$Z[e^{in\pi/2}] = \frac{z}{z-i} \cdot \frac{z+i}{z+i} = \frac{z^2 + iz}{z^2+1}$ — 2m

i.e. $Z\left[\cos\left(\frac{n\pi}{2}\right) + i \sin\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1} + i \cdot \frac{z}{z^2+1}$

$\therefore Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1}$ and $Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z}{z^2+1}$ — 2m

$\therefore$ From Eq^n (1)

$Z[U_n] = \frac{1}{\sqrt{2}} \left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1} \right] = \frac{z(z-1)}{\sqrt{2}(z^2+1)}$ — 2m

6

a) $f(\theta) = \frac{2}{\pi}\int_0^\infty F(\alpha)\cos\alpha\theta\, d\alpha$ (1m)

$= \frac{2}{\pi}\int_0^1 (1-\alpha)\cos\alpha\theta\, d\alpha$, since $F(\alpha)=0$ for $\alpha>1$

$= \frac{2}{\pi}\left[(1-\alpha)\frac{\sin\alpha\theta}{\theta} - (-1)\left(\frac{-\cos\alpha\theta}{\theta^2}\right)\right]_0^1$ (2m)

$= \frac{2}{\pi\theta}\left[(1-\alpha)\sin\alpha\theta\right]_0^1 - \frac{2}{\pi\theta^2}\left[\cos\alpha\theta\right]_0^1$

$= \frac{2}{\pi\theta}[0-0] - \frac{2}{\pi\theta^2}[\cos\theta - 1]$

$f(\theta) = \frac{2(1-\cos\theta)}{\pi\theta^2} = \frac{4\sin^2\theta/2}{\pi\theta^2}$ (1m)

Hence $\int_0^\infty \frac{4\sin^2(\theta/2)}{\pi\theta^2}\cos\alpha\theta \cdot d\theta = F(\alpha) = \begin{cases} 1-\alpha & 0\le\alpha\le1 \\ 0 & \alpha>1 \end{cases}$

$\therefore \int_0^\infty \frac{\sin^2(\theta/2)}{(\theta/2)^2}\cos\alpha\theta\, d\theta = \pi F(\alpha)$. putting $\theta/2 = t$ $\quad \theta = 2t$, $d\theta = 2dt$

ie $\int_0^\infty \frac{\sin^2 t}{t^2}\cdot\cos\alpha\theta\, \cancel{d\theta}\, 2dt = \pi F(\alpha)$, put $\alpha=0$

$$\boxed{\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}}$$ (2m)

6M

b) Let $\bar{u}(z) = \frac{2z^2+3z}{(z+2)(z-4)}$

$\frac{\bar{u}(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$ (1m)

$2z+3 = A(z-4) + B(z+2)$ (2m)

put $z=-2 \Rightarrow A = 1/6$

put $z = 4 \Rightarrow B = 11/6$.

$\therefore \frac{\bar{u}(z)}{z} = \frac{1/6}{z+2} + \frac{11/6}{z-4}$ (2m)

$\bar{u}(z) = \frac{1}{6}\left[\frac{z}{z+2} + \frac{11z}{z-4}\right]$ Taking inverse Z-Transform

$Z^{-1}[\bar{u}(z)] = \frac{1}{6}Z^{-1}\left[\frac{z}{z+2} + \frac{11z}{z-4}\right]$

$u_n = \frac{1}{6}\left[(-2)^n + 11(4)^n\right]$ (2m)

7M

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6c) Taking z-transform on both sides

$Z[y_{n+2}] - 4Z[y_n] = Z[0]$

$z^2[Y(z) - y_0 - y_1 z^{-1}] - 4Y(z) = 0$ — (1m)

$[z^2 - 4]Y(z) - 2z = 0$

$Y(z) = \frac{2z}{z^2-4}$ — (1m)

$\frac{Y(z)}{z} = \frac{2}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$

ie $2 = A(z+2) + B(z-2)$

put $z = -2 \Rightarrow B = -1/2$

put $z = 2 \Rightarrow A = 1/2$ — (2m)

$\frac{Y(z)}{z} = \frac{1/2}{z-2} - \frac{1/2}{z+2}$

$Y(z) = \frac{1}{2}\left[\frac{z}{z-2} - \frac{z}{z+2}\right]$ — (2m)

Taking inverse z-transform

$z^{-1}[Y(z)] = \frac{1}{2} z^{-1}\left[\frac{z}{z-2} - \frac{z}{z+2}\right]$

$y_n = \frac{1}{2}[(2)^n - (-2)^n]$ — (1m)

7M

7

a) Given $\frac{dy}{dx} = x^2 y - 1$, $x_0 = 0$, $y_0 = 1$

$\boxed{y_1(0) = -1}$ — (1m)

$y_2 = x^2 y_1 + 2xy$ $\boxed{y_2(0) = 0}$

$y_3 = x^2 y_2 + 2xy_1 + 2y + 2xy_1$ $\boxed{y_3(0) = 2}$ — (2m)

$y_4 = x^2 y_3 + 2xy_2 + 2xy_2 + 2y_1 + 2y_1 + 2y_1 + 2xy_2$

$y_4(0) = -2-2-2 = -6 \Rightarrow \boxed{y_4(0) = -6}$ — (1m)

∴ By Taylor's series method.

$y = y_0 + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \cdots$ — (1m)

$y = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!} \times (-6)$

$y = 1 - x + \cancel{x^2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ — (1m)

6M

At $x = 0.1$

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4}$$

$\boxed{y(0.1) = 0.9003}$ — 1m

b) Given $\frac{dy}{dx} = x^2 + y$; $x_0 = 0$ $y_0 = 1$, $h = 0.05$.

I st stage: $x_1 = x_0 + h = 0 + 0.05 \Rightarrow \boxed{x_1 = 0.05}$ — 1m

By Euler's formula

$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.05 f(0,1) = 1 + 0.05[0^2 + 1]$

$\boxed{y_1^{(0)} = 1.05}$ — 2m

By modified Euler's formula

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$ — 1m

$= 1 + \frac{0.05}{2}[1 + f(0.05, 1.05)]$

$= 1 + 0.025[1 + \{0.05^2 + 1.05\}] = 1.05131$.

$\boxed{y_1^{(2)} = 1.05131}$ — ~~2m~~

$\therefore \boxed{y(0.05) = 1.05131}$ — 1m

II stage: Let $x_0 = 0.05$, $y_0 = 1.05131$

$y_1^{(0)} = 1.05131 + 0.05[0.05^2 + 1.05131]$

$\boxed{y_1^{(0)} = 1.1040}$ — 1m

$y_1^{(1)} = 1.05131 + 0.025[1.053181 + \{0.1^2 + 1.1040\}]$

$y_1^{(1)} = 1.1055$

$y_1^{(2)} = 1.1055$

$\therefore \boxed{y(0.1) = 1.1055}$ — 01

c)

x	y	$y' = x - y^2$
0	0	0
0.2	0.02	0.1996
0.4	0.0795	0.3936
0.6	0.1762	0.5689
0.8	?	

1m

By Milne's Method

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 0 + \frac{4(0.2)}{3}[2(0.1966) - 0.3937 + 2(0.5689)]$$

$$= 0.3049 \quad \text{(2m)}$$

Now $y_4' = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$

$$y_4^{(c)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.707]$$

$$y_4^{(c)} = 0.3046 \quad \text{(2m)}$$

$$y_4' = x_4 - y_4^2 = 0.8 - (0.3046)^2 = 0.7072$$

$$y_4^{(1)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7072] = 0.3046$$

$$\boxed{y(0.8) = 0.3046} \quad \text{(2m)} \qquad \text{(7m)}$$

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8 a) I stage $f(x_0, y_0) = 0 + \sqrt{1} = 1$ $\quad x_1 = x_0 + h = 0.2$

By Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0) = 1 + 0.2(1) = 1.2 \quad \text{(1m)}$$

By Modified Euler's formula.

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})] = 1.2295$$

$$= 1 + \frac{0.2}{2}[1 + 0.2 + \sqrt{1.2}] = 1.2309$$

$$y_1^{(2)} = 1 + 0.1[1.2 + \sqrt{1.2295}] \qquad \boxed{y(0.2) = 1.2309} \quad \text{(2m)}$$

$$y_1^{(3)} = 1.2309$$

II stage: $x_0 = 0.2$, $y_0 = 1.2309$ $\quad x_1 = 0.4$.

$$y_1^{(0)} = 1.2309 + 0.2(1.3095) = 1.4928 \quad \text{(1m)}$$

By Modified Eulers method.

$$y_1^{(1)} = 1.2309 + \frac{0.2}{2}[1.3095 + 0.4 + \sqrt{1.4928}] = 1.524$$

$$y_1^{(2)} = 1.2309 + 0.1[1.7095 + \sqrt{1.524}] = 1.5254.$$

$$y_1^{(3)} = 1.2309 + 0.1[1.7095 + \sqrt{1.5254}] = 1.5254.$$

Thus $\boxed{y(0.4) = 1.5254}$ (2m) $\qquad$ 6M

b) $f(x, y) = 3x + y/2$, $x_0 = 0$, $y_0 = 1$ $h = 0.2$

$k_1 = hf(x_0, y_0) = 0.2 f(0,1) = 0.2\left[3\times0 + \frac{1}{2}\right] = 0.1$ — (1m)

$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = 0.2 f(0.1, 1.05) = 0.2\left[3\times0.1 + \frac{1.05}{2}\right] = 0.165$ (1m)

$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = 0.2 f(0.1, 1.0825) = 0.2\left[3\times0.1 + \frac{1.0825}{2}\right] = 0.16825$ (1m)

$k_4 = hf(x_0+h, y_0+k_3) = 0.2 f(0.2, 1.16825) = 0.2\left[3\times0.2 + \frac{1.16825}{2}\right] = 0.236825$ (1m)

$y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ — (1m)

$y(0.2) = 1 + \frac{1}{6}\left[0.1 + (2\times0.165) + (2\times0.16825) + 0.236825\right]$

$\boxed{y(0.2) = 1.1672}$ — (2m) 7M

c) Given $\frac{dy}{dx} = (1+y)x^2$

x	y	$y' = x^2(1+y)$
$x_0 = 1$	$y_0 = 1$	$y_0' = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$y_1' = 2.70193$
$x_2 = 1.2$	$y_2 = 1.548$	$y_2' = 3.6691$
$x_3 = 1.3$	$y_3 = 1.979$	$y_3' = 5.0345$
$x_4 = 1.4$	$y_4 = ?$	

(1m)

$y_4^{(P)} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$

$= 1.979 + \frac{0.1}{24}\left[(55\times5.0345) - (59\times3.6691) + (37\times2.70193) - (9\times2)\right]$

$\boxed{y_4^{(P)} = 2.57230}$ — (2M)

$y_4' = x_4^2(1 + y_4^{(P)}) = (1.4)^2\times[1+2.57230] = 7.001$

$y_4^{(C)} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$ — (1m)

$= 1.979 + \frac{0.1}{24}\left[9\times7.001 + (19\times5.0345) - (5\times3.6691) + 2.70193\right]$

$y_4^{(C)} = 2.5749$ — (2m)

$y_4' = x_4^2(1 + y_4^{(C)}) = (1.4)^2[1 + 2.5749] = 7.006$

$y_4^{(C)} = 2.5749 \Rightarrow \boxed{y(1.4) = 2.5749}$ — (1m)

7M

9

a) putting $\frac{dy}{dx} = z$; $y'' = \frac{dz}{dx}$.

$$\frac{dz}{dx} = xz + y; \quad y(0) = 1, \ z(0) = 0$$

$f(x, y, z) = z$, $g(x, y; z) = xz + y$, $x_0 = 0$, $y_0 = 1$, $z_0 = 0$

and $h = 0.2$

$K_1 = hf(x_0, y_0, z_0) = 0.2 f(0, 1, 0) = 0 \Rightarrow \boxed{K_1 = 0}$

$l_1 = 0.2 g(0, 1, 0) = 0.2 \times [0.0 \times 0.0 + 1] = 0.2$ (1m)

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}) = 0.2 f(0.1, 1, 0.1) = 0.02$

$l_2 = hg(x_0 + h/2, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = 0.2 g(0.1, 1, 0.1) = 0.202$ — (1m)

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = 0.2 f(0.1, 1.01, 0.101) = 0.0202$

$l_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = 0.2 g(0.1, 1.01, 0.101) = 0.204$ — (1m)

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2 f(0.2, 1.0202, 0.204) = 0.0408$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2 g(0.2, 1.0202, 0.204) = 0.2122$ (1m)

$y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0 + (2 \times 0.02) + (2 \times 0.0202) + 0.204)$

$\boxed{y(0.2) = 1.0202}$ — (1m)

$z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= 0 + \frac{1}{6}[0.2 + (2 \times 0.202) + (2 \times 0.204) + 0.2122)]$

6M

$\boxed{z(0.2) = 0.204}$ — (1m)

b) Consider $\delta f = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx = \delta I$ — (1m)

By the variational Eq^n $\delta I = 0$

$\therefore \int_{x_1}^{x} \left(\frac{\partial f}{\partial y} \cdot \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx = 0$

since $\delta y' = \frac{d}{dx}(\delta y)$

$\therefore \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} \cdot \delta y + \frac{\partial f}{\partial y'} \cdot \frac{d}{dx}(\delta y) \right\} dx$ — (2m) i.e. $\int_{x_1}^{x_2} \frac{\partial f}{\partial y} \cdot \delta y \, dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} , \frac{d}{dx}\left(\delta y\right) dx = 0$ — (1m)

(1) (2)

Integrating by parts.

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y} \delta y \, dx + \left[\left(\frac{\partial f}{\partial y'} \cdot (\delta y) \right)_{x_1}^{x_2} - \int_{x_1}^{x_2} \left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \delta y \cdot dx \right] = 0$$ — (1m)

since $\delta y = 0$ is the only condition at $x = x_1$ & x_2 'therefore second term is zero

$\therefore \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \delta y \, dx = 0$ for $x_1 < x < x_2$ $\delta y \neq 0$ — (1)

$\therefore E: \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0}$ — (1m)

7m

c) Let $y = y(x)$ be a curve joining two points $P(x_1, y_1)$ & $Q(x_2, y_2)$ in the xoy plane.

We know that the arc length between the pts. P and Q is given by,

$$S = \int_{x_1}^{x_2} \frac{ds}{dx} \cdot dx = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2} \, dx$$ — (1m)

i.e. $I = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx.$ — (1m)

Let $f(x, y, y') = \sqrt{1 + y'^2}$ by Euler's eqn.

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0$ — (1m)

$0 - \frac{d}{dx}\left[\frac{2y'}{2\sqrt{1 + y'^2}} \right] = 0$

or $\frac{d}{dx}\left[\frac{y'}{\sqrt{1 + y'^2}} \right] = 0$ — (2m)

$y'' \sqrt{1 + y'^2} - y' \cdot \frac{2y'y''}{2\sqrt{1 + y'^2}} = 0$

$y''(1 + y'^2) - y''y'^2 = 0$ or $y'' = 0$

Integrating twice $\boxed{y = ax + b}$ — (2m)

which is a straight line.

7M

10

a) Let $I = \int_0^1 [y'^2 + 12xy]\, dx$

Let $f(x, y, y') = (y')^2 + 12xy$ By Euler's Eqn — (1m)

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow 12x - \frac{d}{dx}(2y') = 12x - 2y'' = 0.$

or $y'' = 6x$ integrating. — (2m)

$y' = 3x^2 + c$ Integrating again

$y = x^3 + c_1 x + c_2$ — (2m)

Using the Condn. $y(0) = 0,\ y(1) = 1.$ we obtain $c_1 = 0,\ c_2 = 0$

Thus $\boxed{y = x^3}$ is the required curve — (1m) 6m

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b) putting $y' = z$ we obtain

$\cancel{y''} = \frac{dz}{dx} = 2x + \frac{z}{2}$

$z_0' = 3,\ z_1' = 2x_1 + \frac{z_1}{2} = 3.3589,\quad z_2' = 3.7362,\quad z_3' = 4.1328$ (1m)

$$\left.\begin{aligned} y_4^{(P)} &= y_0 + \frac{4h}{3}[2z_1 - z_2 + 2z_3] \\ z_4^{(P)} &= z_0 + \frac{4h}{3}[2z_1' - z_2' + 2z_3'] \end{aligned}\right\} \text{— (1m)}$$

$y_4^{(P)} = 2 + \frac{4\times 0.1}{3}[(2\times 2.3178) - 2.6725 + (2\times 3.0657)]$

$\boxed{y_4^{(P)} = 3.0792}$ — (1m)

$z_4^{(P)} = 2 + \frac{4\times 0.1}{3}[(2\times 3.3589) - 3.7362 + (2\times 4.1328)]$

$\boxed{z_4^{(P)} = 3.4996}$ — (1m)

$$\left.\begin{aligned} y_4^{(C)} &= y_2 + \frac{h}{3}[z_2 + 4z_3 + z_4] \\ z_4^{(C)} &= z_2 + \frac{h}{3}[z_2' + 4z_3' + z_4'] \end{aligned}\right\} \text{— (1m)}$$

We have $z_4' = 2x_4 + \frac{z_4}{2} = 4.5498$

$y_4^{(C)} = 2.4649 + \frac{0.1}{3}[2.6725 + (4\times 3.0657) + 3.4996] = 3.0793$

$z_4^{(C)} = 2.6725 + \frac{0.1}{3}[3.7362 + (4\times 4.1328) + 4.5498] = 3.4992$

(1m)

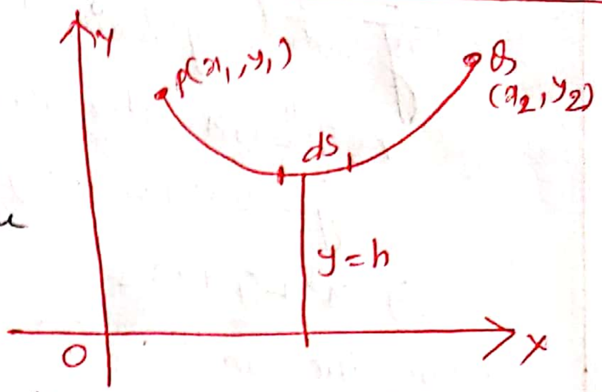
7m

Applying again Corrector formula.

$\boxed{y(1\cdot 4) = 3\cdot 0794}$ —— (1m)

[7m]

c) Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two fixed points on the hanging cable. Let 'ρ' be the density (mass/unit length) of the cable. ρds is the mass of the Element. If 'g' is the accleration due to gravity then the potential energy of the element is $(m.g.h)$ ie $(\rho ds).g.y$.



$\therefore$ Total potential Energy of the cable is given by

$$T = \int_P^Q (\rho ds)\, g.y.dx = \int_{x_1}^{x_2} \rho.g y.\frac{ds}{dx}\,dx$$ —— (2m)

Here $f(x,y,y') = (\rho g) y\sqrt{1+y'^2} = y\sqrt{1+y'^2}$

From Euler's Eqn

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

$$y\sqrt{1+y'^2} - y'.\frac{y}{2\sqrt{1+y'^2}}.2y' = c.$$ —— (2m)

ie $\dfrac{y(1+y'^2) - yy'^2}{\sqrt{1+y'^2}} = c$ or $\dfrac{y}{\sqrt{1+y'^2}} = c.$

ie $y^2 = c^2(1+y'^2)$ or $y'^2 = \dfrac{y^2-c^2}{c^2}$ —— (1m)

$y' = \dfrac{\sqrt{y^2-c^2}}{c}$ or $\dfrac{dy}{\sqrt{y^2-c^2}} = \dfrac{1}{c}\,dx$ Integrating

$\cosh^{-1}(y/c) = \dfrac{x}{c} + k_1$

$y/c = \cosh\left(\dfrac{x}{c} + k_1\right)$ —— (1m)

Thus $y = c\cosh\left[\dfrac{x+a}{c}\right]$ where $a = k_1 c$. —— (1m)

This is a Catenary

—— END ——

[7M]

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