CBCS SCHEME

USN | | | | | | | | | | |

18MATDIP31

Third Semester B.E. Degree Examination, Jan./Feb. 2021
Additional Mathematics – I

Time: 3 hrs. Max. Marks: 100

Note: Answer any FIVE full questions, choosing ONE full question from each module.

Module-1

1 a. Prove that $(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n=2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$. (08 Marks)

b. Express $1-i\sqrt{3}$ in the polar form and hence find its modulus and amplitude. (06 Marks)

c. Find the argument of $\frac{1+\sqrt{3}i}{1-\sqrt{3}i}$. (06 Marks)

OR

2 a. If $\vec{A}=4\hat{i}+3\hat{j}+\hat{k}$ and $\vec{B}=2\hat{i}-\hat{j}+2\hat{k}$ find a unit vector $\hat{N}$ perpendicular to both $\vec{A}$ and $\vec{B}$ such that $\vec{A}$, $\vec{B}$ and N from a right handed system. (08 Marks)

b. If $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$ then show that $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are orthogonal. (06 Marks)

c. Show that the position vectors of the vertices of a triangle $\vec{A}=3(\sqrt{3}\,\hat{i}-\hat{j})$, $\vec{B}=6\hat{i}$ and $C=3(\sqrt{3}\hat{i}+j)$ form an isosceles triangle. (06 Marks)

Module-2

3 a. Obtain the Maclaurin series expansion of $\log\sec x$ upto to the terms containing x^6. (08 Marks)

b. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that $xu_x+yu_y=\sin 2u$. (06 Marks)

c. If $u=f(x-y,y-z,z-x)$, show that $u_x+u_y+u_z=0$. (06 Marks)

OR

4 a. Prove that $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ldots$ by using Maclaurin's series notation. (08 Marks)

b. Using Euler's theorem, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u\log u$. If $u=e^{\frac{x^2y^2}{x+y}}$. (06 Marks)

c. If $u=x+y$, $v=y+z$, $w=z+x$, find $J\left(\frac{u,v,w}{x,y,z}\right)$. (06 Marks)

Module-3

5 a. A particle moves along the curve $\vec{r}=\cos 2t\,\hat{i}+\sin 2t\,\hat{j}+t\,\hat{k}$, find the velocity and acceleration at $t=\frac{\pi}{8}$ along $\sqrt{2}\,\hat{i}+\sqrt{2}\,\hat{j}+\hat{k}$. (08 Marks)

b. Find the unit normal to the surface, $xy+x+zx=3$ at $(1,1,1)$. (06 Marks)

c. Find the constant 'a' such that the vector field $\vec{F}=2xy^2z^2\hat{i}+2x^2yz^2\hat{j}+ax^2y^2z\hat{k}$ is irrotational. (06 Marks)

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

OR

6 a. If $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ show that $\vec{F}.\text{curl}\vec{F} = 0$. (08 Marks)

b. If $\phi(x,y,z) = xy^2 + yz^3$, find $\nabla\phi$ & $|\nabla\phi|$ at (1, -2, -1) (06 Marks)

c. Show that vector field $\vec{F} = \left[\frac{x\hat{i} + y\hat{j}}{x^2 + y^2}\right]$ is solenoidal. (06 Marks)

Module-4

7 a. Obtain a reduction for $\int_0^{\frac{\pi}{2}} \sin^n x dx$ (n > 0). (08 Marks)

b. Evaluate $\int_0^1 \frac{x^9}{\sqrt{1-x^2}} dx$. (06 Marks)

c. Evaluate $\iint_R xydxdy$ where R is the first quadrant of the circle $x^2 + y^2 = a^2$, $x \geq 0$, $y \geq 0$. (06 Marks)

OR

8 a. Obtain a reduction formula for $\int_0^{\frac{\pi}{2}} \cos^n xdx$, (n > 0). (08 Marks)

b. Evaluate $\int_0^{2a} x^2\sqrt{2ax - x^2}dx$. (06 Marks)

c. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dydxdz$ (06 Marks)

Module-5

9 a. Solve $\frac{dy}{dx} + y\cot x = \sin x$. (08 Marks)

b. Solve $\cos x \sin ydx + \cos y \sin xdy = 0$. (06 Marks)

c. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$. (06 Marks)

OR

10 a. Solve : $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ (08 Marks)

b. Solve : $\frac{dy}{dx} + \frac{y}{x} = y^2 x$. (06 Marks)

c. Solve : $\sqrt{1-y^2}dx = (\sin^{-1} y - x)dy$ (06 Marks)

* * * * *

Module - I

1a) P.T $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{n\theta}{2}\right)$

Soln L.H.S $= (1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$= (2\cos^2\theta/_2 + i\,2\sin\theta/_2\cos\theta/_2)^n +$

$(2\cos^2\theta/_2 - i\,2\sin\theta/_2\cos\theta/_2)^n$ — (3m)

$= 2^n(\cos\theta/_2)^n\left\{(\cos\theta/_2 + i\sin\theta/_2)^n + (\cos\frac{\theta}{2} - i\sin\frac{\theta}{2})^n\right\}$

$= 2^n\cos^n\theta/_2\left\{\cos n\theta/_2 + \cancel{i\sin\frac{n\theta}{2}} + \cos\frac{n\theta}{2} - \cancel{i\sin\frac{n\theta}{2}}\right\}$

(3m)

$= 2^n\cos^n\frac{\theta}{2}\left\{2\cos\frac{n\theta}{2}\right\}$

$= 2^{n+1}\cos^n\frac{\theta}{2}\cos\left(\frac{n\theta}{2}\right) = \text{RHS}$ — (2m)

L.H.S = R.H.S

1b) Express $1-i\sqrt{3}$ in polar form and hence find its modulus and amplitude.

Sol: Let $1-i\sqrt{3} = r\cos\theta + i r\sin\theta$. — (1m)

$\Rightarrow \quad r\cos\theta = 1 \qquad r\sin\theta = -\sqrt{3}$

$r^2\cos^2\theta = 1 \qquad r^2\sin^2\theta = 3$

$r^2\cos^2\theta + r^2\sin^2\theta = 1+3$

$r^2(\cos^2\theta + \sin^2\theta) = 4$

$r^2 = 4 \Rightarrow r = 2.$ —(2m)

$\frac{r\sin\theta}{r\cos\theta} = \frac{-\sqrt{3}}{1} \Rightarrow \tan\theta = -\sqrt{3} \Rightarrow \theta = \tan^{-1}(-\sqrt{3})$

$\theta = -\frac{\pi}{3}$ or $\theta = -60^\circ$ here r is magnitude and θ is amplitude.

—(3m)

1c) Find the argument of $\frac{1+\sqrt{3}i}{1-\sqrt{3}i}$

Sol: $\frac{1+\sqrt{3}i}{1-\sqrt{3}i} \times \frac{1+\sqrt{3}i}{1+\sqrt{3}i} = \frac{(1+\sqrt{3}i)^2}{1^2 - (\sqrt{3})^2 i^2}$ —(1m)

$= \frac{1 + 3i^2 + i2\sqrt{3}}{1-(3)(-1)} = \frac{1-3+i2\sqrt{3}}{1+3}$ $\because i^2 = -1$

$= \frac{-2+i2\sqrt{3}}{4} = -\frac{\cancel{2}}{\cancel{4}_2} + i\frac{\cancel{2}\sqrt{3}}{\cancel{4}_2}.$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$ —(3m)

$a+ib = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow a = -\frac{1}{2}$ $b = \frac{\sqrt{3}}{2}$ —(1m)

Amplitude or Argument $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\theta = \tan^{-1}\left\{\frac{\sqrt{3}}{\cancel{2}} \cdot \left(\frac{-\cancel{2}}{1}\right)\right\} = -\tan^{-1}\sqrt{3} = -\frac{\pi}{3}$

$\theta = -60^\circ$ —(1m)

2a) If $\vec{A} = 4\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}$ find a unit vector $\hat{N}$ perpendicular to both $\vec{A}$ and $\vec{B}$ such that $\vec{A}$, $\vec{B}$ and N form a right handed system.

Sol: Unit vector N perpendicular to $\vec{A}$ and $\vec{B}$

$$\hat{N} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} \quad \text{—(1m)}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7i - 6j - 10k \quad \text{—(4m)}$$

$$|\vec{A} \times \vec{B}| = \sqrt{7^2 + (-6)^2 + (-10)^2} = \sqrt{49 + 36 + 100} = \sqrt{185} \quad \text{—(2m)}$$

$$\Rightarrow \hat{N} = \frac{7i - 6j - 10k}{\sqrt{185}} \quad \text{—(2m)}$$

2b) If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ then show that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal.

Sol: $\vec{a} + \vec{b} = 4\hat{i} + \hat{j} - k$ (2m), $\vec{a} - \vec{b} = -2\hat{i} + 3\hat{j} - 5k$ (2m)

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (4)(-2) + (1)(3) + (-1)(-5)$$

$$= -8 + 3 + 5 = \cancel{-8} + \cancel{8} = 0 \quad \text{—(2m)}$$

$\Rightarrow (\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal.

2() Show that the position vectors of the vertices of a triangle $\vec{A} = 3(\sqrt{3}i - j)$ $\vec{B} = 6i$ and $\vec{C} = 3(\sqrt{3}\,i + j)$ form an isosceles triangle.

Sol?

$|\vec{A}| = \sqrt{(3\sqrt{3})^2 + (-3)^2} = \sqrt{27+9} = \sqrt{36} = 6$ — (1m)

$|\vec{B}| = \sqrt{6^2 + 0 + 0} = \sqrt{36} = 6$ — (1m)

$|\vec{C}| = \sqrt{(3\sqrt{3})^2 + 3^2} = \sqrt{27+9} = 6$ — (1m)

we observed that $|\vec{a}| = |\vec{b}| = |\vec{c}| = 6$ — (1m)

Thus we conclude that the triangle is an Equilateral triangle and also isosceles tringle. — (1m)

3a) Maclaurin's series Expansion of log sec x

$y(x) = \log\sec x$

Maclaurin's series Expansion is given by

$y(x) = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \cdots$ (1m)

$y(0) = \log\sec 0 \Rightarrow y(0) = 0$

$y_1(x) = \frac{\sec x \tan x}{\sec x} \Rightarrow y_1(x) = \tan x,\ y_1(0) = 0$

$y_2(x) = \sec^2 x = 1 + \tan^2 x,\ y_2(0) = 1$

$y_2(x) = 1 + y_1^2 \Rightarrow y_3(x) = 2y_1y_2$

$y_3(0) = 2(0)(1) = 0$

$y_4(x) = 2[y_1y_3 + y_2y_2] = 2[y_1y_3 + y_2^2]$

$y_4(0) = 2[(0)(0) + 1^2] = 2$

$y_5(x) = 2[y_1y_4 + y_3y_2 + 2y_2y_3]$

$y_5(x) = 2[0 + 0 + 0] = 0$

$y_5(x) = 2[y_1y_4 + 3y_2y_3]$

$y_6(x) = 2[y_1y_5 + y_4y_2 + 3(y_2y_4 + y_3 \cdot y_3)]$

$y_6(x) = 2[0 + (2)(1) + 3(1 \cdot 2 + 0)] = 16$ (6m)

$\Rightarrow y(x) = 0 + 0 + \frac{x^2}{2}(1) + 0 + \frac{x^4}{24}(2) + 0 + \frac{x^6(16)}{720} + \cdots$

$\log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$ // (1m)

3 b) $u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$ P.T $x u_x + y u_y = \sin 2u$.

Sol: $\tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left[1+y^3/x^3\right]}{x\left[1-y/x\right]} = \frac{x^2\left[1+y^3/x^3\right]}{\left[1-y/x\right]}$

$\tan u = x^2 g(y/x)$ — (1m)

$\Rightarrow$ tan u is homogeneous of degree 2

Applying Euler's theorem for the function tan u taking $n=2$ — (1m)

$x \frac{\partial}{\partial x} \tan u + y \frac{\partial}{\partial y} \tan u = 2 \tan u$

$x \frac{\partial u}{\partial x} \sec^2 u + y \frac{\partial u}{\partial y} \sec^2 u = 2 \tan u$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u}$

$= 2 \frac{\sin u}{\cancel{\cos u}} \cos^{\cancel{2}} u = \sin 2u$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$ — (4m)

3 c) $u = f(x-y, y-z, z-x)$ S.T $u_x + u_y + u_z = 0$

Put $x-y = p$ $y-z = q$ $z-x = r$

$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$

$u_x = \frac{\partial u}{\partial p}(1) + 0 + \frac{\partial u}{\partial r}(-1) = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}$ — (2m)

iii) $u_y = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}$, $u_z = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}$ —(2m)

$u_x + u_y + u_z = \cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial q}} - \cancel{\frac{\partial u}{\partial q}} + \cancel{\frac{\partial u}{\partial r}}$

$u_x + u_y + u_z = 0.$ —(2m)

4a) $y(x) = \log(1+x)$

Maclaurin's series expansion is given by

$y(x) = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + ---$ —(2m)

$y(0) = \log(1+0) = \log 1 = 0$

$y_1(x) = \frac{1}{1+x} \Rightarrow y_1(0) = 1$

$y_2(x) = \frac{-1}{(1+x)^2} \Rightarrow y_2(0) = -1$

$y_3(x) = \frac{2}{(1+x)^3} \Rightarrow y_3(0) = 2$

$y_4(x) = \frac{-6}{(1+x)^4} \Rightarrow y_4(0) = -6$ —(4m) etc

$\Rightarrow y(x) = 0 + \frac{x}{1}(1) + \frac{x^2(-1)}{2} + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + ---$ —(2)

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ---$ —(2) //

4 b) If $u = e^{x^2y^2/x+y}$ P.T $xu_x + yu_y = 3u\log u$

Sol:

$$u = e^{x^2y^2/x+y}$$

$$\log u = \frac{x^2y^2}{x+y} = \frac{\frac{x^2 y^2 x^2}{x^2}}{x\left(1+\frac{y}{x}\right)}$$

$$\log u = \frac{x^4\left(\frac{y}{x}\right)^2}{\cancel{x}\left(1+\frac{y}{x}\right)} = x^3 g\left(\frac{y}{x}\right) \quad \text{— (1m)}$$

$\log u$ is homogeneous of degree 3

Applying Euler's thm for the fun $\log u$ — (1m)

$$x\frac{\partial}{\partial x}\log u + y\frac{\partial}{\partial y}\log u = 3\log u$$

$$x\cdot\frac{1}{u}\frac{\partial u}{\partial x} + y\cdot\frac{1}{u}\frac{\partial u}{\partial y} = 3\log u$$

$$\frac{1}{u}\left\{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right\} = 3\log u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u$$

$$xu_x + yu_y = 3u\log u. \quad \text{— (4m)}$$

4.c) If $u = x+y$ $v = y+z$ $w = z+x$ find $J\left(\frac{u,v,w}{x,y,z}\right)$

Sol: $J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$ (2m)

(2m)

$= 1(1-0) - 1(0-1) + 0$

$J\left(\frac{u,v,w}{x,y,z}\right) = 1+1 = 2$ — (2m)

Module - 3

5(a) If $\vec{r} = \cos 2t\,\hat{i} + \sin 2t\,\hat{j} + t\hat{k}$ find the velocity and acceleration at $t = \frac{\pi}{8}$ along $\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}$

Soln $\vec{r} = \cos 2t\,\hat{i} + \sin 2t\,\hat{j} + t\,\hat{k}$

$$\frac{d\vec{r}}{dt} = \vec{V} = -2\sin 2t\,\hat{i} + 2\cos 2t\,\hat{j} + \hat{k}$$

$$\vec{A} = \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d\vec{V}}{dt} = -4\cos 2t\,\hat{i} - 4\sin 2t\,\hat{j} + 0$$

At $t = \pi/4$

$$\vec{V} = -2 \cdot \frac{1}{\sqrt{2}}\,\hat{i} + 2 \cdot \frac{1}{\sqrt{2}}\,\hat{j} + \hat{k} = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}$$ (2m)

$$\vec{A} = -4 \cdot \frac{1}{\sqrt{2}}\,\hat{i} + 4 \cdot \frac{1}{\sqrt{2}}\,\hat{j} = -2\sqrt{2}\,\hat{i} - 2\sqrt{2}\,\hat{j}$$ (2m)

Unit vector in the given direction $\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}$ is

$$\hat{n} = \frac{\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}}{\sqrt{2+2+1}} = \frac{\sqrt{2}\,i + \sqrt{2}\,j + k}{\sqrt{5}}$$ (2m)

Velocity and Accelaration along $\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}$ is given by

$$\vec{V} \cdot \hat{n} = -\sqrt{2}\,i + \sqrt{2}\,j + k \frac{(\sqrt{2}\,i + \sqrt{2}\,j + k)}{\sqrt{5}} = \frac{-2+2+1}{\sqrt{5}}$$

$$\vec{V}\,\hat{n} = \frac{1}{\sqrt{5}}$$ # — (1m)

$$\vec{A}\,\hat{n} = -2\sqrt{2}\,i - 2\sqrt{2}\,j \frac{(\sqrt{2}\,i + \sqrt{2}\,j + k)}{\sqrt{5}} = \frac{-4-4+0}{\sqrt{5}}$$

$$\vec{A}\,\hat{n} = -8/\sqrt{5}$$ # — (1m)

5 b) Find the unit normal to the surface $xy + x + zx = 3$ at $(1,1,1)$

Sol: Let $\phi = xy + x + zx$

$\nabla\phi$ is the vector normal to the surface

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k \quad \text{— (1m)}$$

$$\nabla\phi = (y+1+z)\, i + (x)\, j + x k$$

$$\nabla\phi_{(1,1,1)} = 3i + j + k \quad \text{— (2m)}$$

Unit normal vector $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$ — (1m)

$$\hat{n} = \frac{3i+j+k}{\sqrt{9+1+1}} = \frac{3i+j+k}{\sqrt{11}} \quad \text{— (2m)}$$

5c) Find the constant 'a' such that the vector field $\vec{F} = 2xy^2z^2 i + 2x^2yz^2 j + a x^2y^2z k$ is irrotational.

Sol: If $\vec{F}$ represents the given vector, we shall find 'a' such that curl $\vec{F} = 0$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2z^2 & 2x^2yz^2 & ax^2y^2z \end{vmatrix}$$

(3m) —

$$i\{ax^2z(2y) - 2x^2y(2z)\}$$

$$-j\{ay^2z(2x) - 2xy^2(2z)\}$$

$$+k\{2yz^2(2x) - 2xz^2(2y)\} = 0 \quad \text{— (2m)}$$

$$i\,2x^2yz(a-2) - j\,2xy^2z(a-2) + k(0) = 0$$

$\nabla \times \vec{F} = 0$ is satisfied when $a = 2$.

Thu $a = 2$ # —(1m)

6a) If $\vec{F} = (x+y+1)\,i + j - (x+y)\,k$ show that

$\vec{F} \cdot \text{curl}\,\vec{F} = 0$

Soln:

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

(3m)

$$i(-1-0) - j(-1-0) + k(0-1) = 0$$

$$-i + j - k = 0 \quad \text{— (2m)}$$

$$\vec{F} \cdot \text{curl}\,\vec{F} = (x+y+1)(-1) + (1)(1) + [-(x+y)][-1]$$

$$= -\cancel{x} - \cancel{y} - \cancel{1} + \cancel{1} + \cancel{x} + \cancel{y}$$

$$= 0 \quad \text{— (3m)}$$

Thus $\vec{F} \cdot \text{cur}\,\vec{F} = 0$

6 b) If $\phi(x, y, z) = xy^2 + yz^3$, find $\nabla\phi$ and $|\nabla\phi|$ at $(1, -2, -1)$

Sol: we have $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$$\nabla\phi = \frac{\partial}{\partial x}[xy^2 + yz^3] i + \frac{\partial}{\partial y}[xy^2 + yz^3] j + \frac{\partial}{\partial z}[xy^2 + yz^3]$$

$$\nabla\phi = y^2 i + [2xy + z^3] j + [3yz^2] \text{ — (3m)}$$

$$\nabla\phi_{(1,-2,-1)} = (-2)^2 i + [2(1)(-2) + (-1)^3] j + 3(-2)(-1)^2 k$$

$$\nabla\phi_{(1,-2,-1)} = 4i + [-4 - 1] j - 6k = 4i - 5j - 6k \text{ — (2m)}$$

$$|\nabla\phi| = \sqrt{4^2 + (-5)^2 + (-6)^2}$$

$$= \sqrt{16 + 25 + 36}$$

$$|\nabla\phi| = \sqrt{77} \text{ —— (1m)}$$

6c) show that vector field $\vec{F} = \left[\dfrac{x\hat{i} + y\hat{j}}{x^2+y^2}\right]$ is solenoidal.

Sol: A vector $\vec{F}$ is solenoidal if $\text{div}\,\vec{F} = 0$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left\{\frac{xi + yj}{x^2+y^2}\right\} \quad \text{— (1m)}$$

$$= \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\left\{\frac{x}{x^2+y^2}i + \frac{y}{x^2+y^2}j\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) + 0 \quad \text{— (2m)}$$

$$= \left\{\frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}\right\} + \left\{\frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}\right\}$$

$$= \left\{\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right\} + \left\{\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right\} \quad \text{— (2m)}$$

$$= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)2} = 0$$

$$= \frac{2\cancel{x^2} - 2\cancel{x^2} + 2\cancel{y^2} - 2\cancel{y^2}}{(x^2+y^2)2} = 0 \quad \text{— (1m)}$$

$\text{div}\,\vec{F} = 0 \Rightarrow \vec{F}$ is solenoidal. //

Module - 4

7a) obtain a reduction for $\int_0^{\pi/2} \sin^n x \, dx$, $n>0$

Sol? $I_n = \int \sin^n x \, dx$

$= \int \sin^{n-1} x \sin x \, dx$ —(1m)

Applying the rule of integration by parts

$I_n = \sin^{n-1} x \int \sin x \, dx - \int (-\cos x)(n-1)\sin^{n-2} x \cos x \, dx$

$I_n = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x (1-\sin^2 x)\, dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx$

$I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$

$I_n \{1 + (n-1)\} = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

$$I_n = \frac{-\sin^{n-1} x \cos x + (n-1) I_{n-2}}{n}$$

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}$$ (4m)

To find $\int_0^{\pi/2} \sin^n x \, dx$

$$I_n = \left[-\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \left(\frac{n-1}{n}\right) I_{n-2}$$ (1m)

$\cos \pi/2 = 0 = \sin 0$

Thus $I_n = \frac{n-1}{n} I_{n-2}$.

* This relation will give us

$$\int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} & \text{if } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even.} \end{cases}$$

(2 m)

7b) Evaluate $\int_0^1 \frac{x^9}{\sqrt{1-x^2}}\, dx$

Sol: $I = \int_0^1 \frac{x^9}{\sqrt{1-x^2}}\, dx$

Put $x = \sin\theta \Rightarrow x^2 = \sin^2\theta$ — (1m)

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$ — (1m)

$dx = \cos\theta\, d\theta$

Put $x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

Put $x = 1 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$

$\theta \rightarrow 0$ to $\pi/2$ — (1 m)

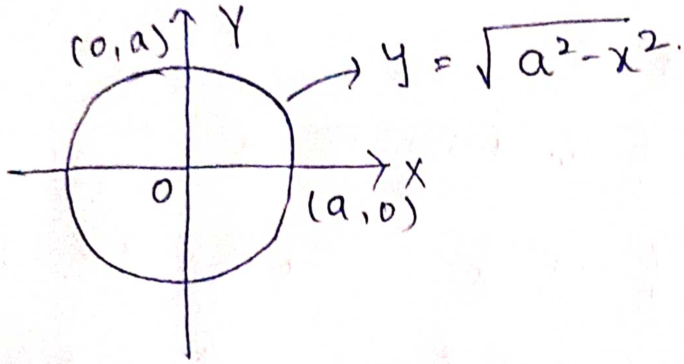
$I = \int_0^{\pi/2} \frac{(\sin\theta)^9}{\cancel{\cos\theta}} \cancel{\cos\theta}\, d\theta = \int_0^{\pi/2} (\sin\theta)^9\, d\theta$ — (1m)

$I = \frac{8}{\cancel{9}_3} \cdot \frac{\cancel{6}^2}{\cancel{7}} \cdot \frac{4}{\cancel{5}} \cdot \frac{2}{3}$ using reduction formula

$I = 128/315$ ## — (2 m)

7c) Evaluate $\iint_R xy\,dx\,dy$ where R is the first quadrant of the circle $x^2+y^2=a^2$; $x\geq0, y\geq0$

Sol:



x varies from 0 to a —(1m)

y varies from 0 to $\sqrt{a^2-x^2}$ —(1m)

$$I=\iint xy\,dx\,dy=\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$I=\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}dx$$

$$I=\int_{x=0}^{a} x\cdot\frac{1}{2}\left[a^2-x^2-0\right]dx$$

$$=\frac{1}{2}\int_0^a (a^2x-x^3)\,dx$$

$$=\frac{1}{2}\left[a^2\cdot\frac{x^2}{2}-\frac{x^4}{4}\right]_0^a$$

$$=\frac{1}{2}\left[\frac{a^2}{2}(a^2-0)-\frac{1}{4}(a^4-0)\right]$$

$$=\frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]=\frac{1}{2}\cdot\frac{a^4}{4}=\frac{a^4}{8}$$ —(4m)

8a) Obtain the reduction formula for $\int_0^{\pi/2} \cos^n x\, dx$, $n>0$

Sol: Let $I_n = \int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx$

Integrating by parts (1m)

$I_n = \cos^{n-1} x \sin x - \int \sin x (n-1) \cos^{n-2} x (-\sin x)\, dx$

$= \cos^{n-1} x \sin x + \int \cos^{n-2} x \sin^2 x\, dx\, (n-1)$

$= \cos^{n-1} x \sin x + \int \cos^{n-2} x (1-\cos^2 x)\, dx\, (n-1)$

$= \cos^{n-1} x \sin x + \int (\cos^{n-2} x\, dx - \cos^n x)(n-1)\, dx$

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\, dx - (n-1) \int \cos^n x\, dx$

$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$

$I_n [1 + (n-1)] I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$I_n = \dfrac{\cos^{n-1} x \sin x}{n} + \dfrac{n-1}{n} I_{n-2}$ —— (1)

(4m)

To find $\int_0^{\pi/2} \cos^n x\, dx$

Let us take

$I_n = \int_0^{\pi/2} \cos^n x\, dx$ ∴ from Eqn (1)

$I_n = \left[\dfrac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \dfrac{n-1}{n} I_{n-2}$ —(1m)

But $\cos \pi/2 = 0 = \sin 0 \Rightarrow I_n = \dfrac{n-1}{n} I_{n-2}$

$\Rightarrow I_n = \begin{cases} \dfrac{n-1}{n} \dfrac{n-3}{n-2} \cdots \cdot \dfrac{2}{3} & \text{if } n \text{ is odd} \\ \dfrac{n-1}{n} \dfrac{n-3}{n-2} \cdots \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \text{if } n \text{ is even.} \end{cases}$

(2m)

8b) Evaluate $\int_0^{2a} x^2\sqrt{2ax-x^2}\,dx$

Sol: $I = \int_0^{2a} x^2\sqrt{2ax-x^2}\,dx$

Put $x = 2a\sin^2\theta \Rightarrow dx = 2a \cdot 2\sin\theta\cos\theta\,d\theta$

$dx = 4a\sin\theta\cos\theta\,d\theta$ — (1m)

$2ax - x^2 = 2a(2a\sin^2\theta) - (2a\sin^2\theta)^2$

$= 4a^2\sin^2\theta - 4a^2\sin^4\theta = 4a^2(\sin^2\theta - \sin^4\theta)$

$= 4a^2\sin^2\theta(1-\sin^2\theta) = 4a^2\sin^2\theta\cos^2\theta$ (1m)

$\sqrt{2ax-x^2} = \sqrt{4a^2\sin^2\theta\cos^2\theta} = 2a\sin\theta\cos\theta.$

$x^2\sqrt{2ax-x^2} = (2a\sin^2\theta)^2 \cdot 2a\sin\theta\cos\theta.$

$= 4a^2\sin^4\theta \cdot 2a\sin\theta\cos\theta$

$= 8a^3\sin^5\theta\cos\theta.$ — (1m)

Put $x = 0 \Rightarrow 0 = 2a\sin^2\theta \Rightarrow \theta = 0$

Put $x = 2a \Rightarrow 2a = 2a\sin^2\theta \Rightarrow \theta = \pi/2$ (1m)

$I = \int_{\theta=0}^{\pi/2} 8a^3\sin^5\theta\cos\theta \cdot 4a\sin\theta\cos\theta\,d\theta$

$= 32a^4\int_0^{\pi/2}\sin^6\theta\cos^2\theta\,d\theta$

Applying reduction formula

$I = \cancel{32}^{\,4}a^4\left\{\frac{(5)(\cancel{3})(1)(1)}{\cancel{8}\cdot\cancel{6}_{2}\cdot\cancel{4}\cdot 2}\cdot\frac{\pi}{2}\right\} = \frac{5a^4\pi}{8\cancel{4}}$

(2m)

8c) Evaluate $\int_{-1}^{1} \int_{0}^{z} \int_{x-2}^{x+2} (x+y+z)\, dy\, dx\, dz$

sol: let $I = \int_{z=-1}^{1} \int_{x=0}^{z} \int_{y=x-2}^{x+2} (x+y+z)\, dy\, dx\, dz$.

$$I = \int_{z=-1}^{1} \int_{x=0}^{z} \left[xy + \frac{y^2}{2} + yz \right]_{x-2}^{x+2} dx\, dz \quad \text{— (1m)}$$

$$I = \int_{z=-1}^{1} \int_{x=0}^{z} \Big\{ x[x+2-(x-2)] + \frac{1}{2}\left[(x+2)^2-(x-2)^2\right] + z[(x+2)-(x-2)] \Big\} dx\, dz \quad \text{— (1m)}$$

$$I = \int_{z=-1}^{1} \int_{x=0}^{z} \Big\{ x[\cancel{x}+2-\cancel{x}+2] + \frac{1}{2}[\cancel{x^2}+4x+\cancel{4}-\cancel{x^2}+4x-\cancel{4}] + z[\cancel{x}+2-\cancel{x}+2] \Big\} dx\, dz$$

$$I = \int_{z=-1}^{1} \int_{x=0}^{z} (4x+4x+4z)\, dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} 4(2x+z)\, dx\, dz. \quad \text{— (2m)}$$

$$= 4\int_{z=-1}^{1} \left[\cancel{2}\cdot\frac{x^2}{\cancel{2}} + zx \right]_0^z dz$$

$$= 4\int_{z=-1}^{1} \Big\{ [z^2-0] + z[z-0] \Big\} dz$$

$$I = 4 \int_{z=-1}^{1} (z^2 + z^2)\, dz \quad \text{—— (1m)}$$

$$= 4 \int_{z=-1}^{1} 2z^2\, dz = 8 \int_{z=-1}^{1} z^2\, dz$$

$$I = 8 \left[\frac{z^3}{2} \right]_{-1}^{1}$$

$$I = \frac{\cancel{8}^{\,4}}{\cancel{2}} \left[1^3 - (-1)^3 \right]$$

$$= 4 \left[1 - (-1) \right] = 4(2)$$

$$I = 8 \quad \text{—— (1m)}$$

Module - 5

9a) Solve $\frac{dy}{dx} + y \cot x = \sin x$

Sol: This is of the form $\frac{dy}{dx} + Py = Q$

where $P = \cot x$ $\quad Q = \sin x$ — (2m)

Hence soln is $y(IF) = \int Q(IF)\,dx + c$

$I.F = e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$ — (2m)

$y(\sin x) = \int \cos x \cdot \sin x\, dx + c$

$= \frac{1}{2}\int 2\sin x \cos x\, dx + c$ — (2m)

$y \sin x = \frac{1}{2}\int \sin 2x\, dx + c$

$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + c$ — (2m)

$y \sin x = -\frac{\cos 2x}{4} + c$

9b) Solve $\cos x \sin y\, dx + \cos y \sin x\, dy = 0$

Sol: We have Exact Differential Eqn

$M = \cos x \sin y$ $\quad N = \sin x \cos y$

$\frac{\partial M}{\partial y} = \cos x \cos y$ $\quad \frac{\partial N}{\partial x} = \cos x \cos y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ — (2m)

Hence sol^n is

$$\int_{y-const} M\,dx + \int N\,dy = c$$

only those terms free from x in N — (1m)

$$\int_{y-const} \cos x \sin y \, dx + \int(0)dy = c \quad \text{—(1m)}$$

$$\sin y \int \cos x \, dx = c$$

$$\sin x \sin y = c. \quad \text{—(2m)}$$

9 c) Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

sol^n: Dividing the Eq^n by y^2 we have

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{1}{x} = x \quad \text{—(1m)}$$

Put $\frac{1}{y} = t$, diff w.r.t x $\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx} \quad \text{—(1m)}$$

$$\Rightarrow -\frac{dt}{dx} + \frac{1}{x}t = x \Rightarrow \frac{dt}{dx} - \frac{t}{x} = -x$$

which linear Eq^n of the form

$\frac{dt}{dx} + Pt = Q$ wher $P = -\frac{1}{x}$ $Q = -x$

Hence solution is

$t\,(IF) = \int Q\,(IF)\,dx + c$ —(1m)

$IF = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x}$

$I.F = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$ —(1m)

$t \cdot \frac{1}{x} = \int (-\cancel{x}) \cdot \frac{1}{\cancel{x}}\,dx + c.$

$\frac{t}{x} = -\int 1\,dx + c$

$\frac{t}{x} = -x + c$

Put $t = \frac{1}{y}$

$\Rightarrow \frac{1}{xy} = -x + c$ —(2m.)

10a) Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Sol: The given Eqn can be put in the form

$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$

$M = y\cos x + \sin y + y \qquad N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 \qquad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$

(3m)

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, The given d.e is exact d.e

Hence solution is $\int\limits_{y\text{-const}} M\,dx + \int N\,dy = C$

only those terms free from x in N (1m)

$\int\limits_{y\text{-const}} (y\cos x + \sin y + y)\,dx + \int 0\,dy = C$ (1m)

$y\sin x + x\sin y + xy = C$ (1m)

10b) solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x.$ (9(c))

10c) solve $\sqrt{1-y^2}\,dx = (\sin^{-1}y - x)\,dy$

Sol: $\frac{dx}{dy} = \frac{\sin^{-1}y - x}{\sqrt{1-y^2}}$

$\frac{dx}{dy} = \frac{\sin^{-1}y}{\sqrt{1-y^2}} - \frac{x}{\sqrt{1-y^2}}$

$$\frac{dx}{dy} + \frac{x}{\sqrt{1-y^2}} = \frac{\sin^{-1}y}{\sqrt{1-y^2}}$$ and this is of the

form $\frac{dx}{dy} + Px = Q$ where $P = \frac{1}{\sqrt{1-y^2}}$

$Q = \frac{\sin^{-1}y}{\sqrt{1-y^2}}$ — (2m)

Hence soln is $x\,(I\,F) = \int Q\,(IF)\,dx + c$

$I.F = e^{\int P dy} = e^{\int 1/\sqrt{1-y^2}\,dy} = e^{\sin^{-1}y}$ — (1m)

$\Rightarrow x\,e^{\sin^{-1}y} = \int \frac{\sin^{-1}y}{\sqrt{1-y^2}}\,e^{\sin^{-1}y}\,dy + c$ — (1m)

Put $\sin^{-1}y = t \Rightarrow \frac{1}{\sqrt{1-y^2}}\,dy = dt$

$\Rightarrow x e^{\sin^{-1}y} = \int t\,e^t\,dt + c$

$x\,e^{\sin^{-1}y} = t\,e^t - e^t + c$ by using integration byparts

$x\,e^{\sin^{-1}y} = \sin^{-1}y\,e^{\sin^{-1}y} - e^{\sin^{-1}y} + c$

$x\,e^{\sin^{-1}y} = e^{\sin^{-1}y}\,[\sin^{-1}y - 1] + c$ — (2m)

Staff Name - Vijaya. C

Dr. Meenal M. Kaliwal
HOD
Department of Mathematics
KLS V.D.I.T., Haliyal

19/3/22
Dean, Academics.