Model Question Paper-I with effect from 2021 (CBCS Scheme)

USN | | | | | | | | | | |

First Semester B.E Degree Examination
Calculus and Differential Equations (21MAT11)

TIME: 03 Hours **Max. Marks: 100**

Note: Answer any **FIVE** full questions, choosing at least **ONE** question from each **MODULE.**

		Module -1	Marks
Q.01	a	With usual notations prove that $\tan\varphi = r\frac{d\theta}{dr}$	06
	b	Find the angle between the curves $r = a(1+\cos\theta)$ *and* $r = a(1-\cos\theta)$	07
	c	Show that the radius of curvature at any point of the cycloid $x = a(\theta = \sin\theta),\ y = a(1-\cos\theta)$ *is* $4a\cos\left(\frac{\theta}{2}\right)$	07
		OR	
Q.02	a	If p be the perpendicular from the pole on the tangent, then show that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$	06
	b	Find the pedal equation of the curve $r^m = a^m(\cos m\theta + \sin m\theta)$	07
	c	Find the radius of curvature of the curve $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$	07
		Module-2	
Q. 03	a	Expand e^{sinx} by Maclaurin's series up to the term containing x^4	06
	b	If $u = f(2x-3y, 3y-4z, 4z-2x)$, show that $6u_x + 4u_y + 3u_z = 0$	07
	c	Examine the function $f(x,y) = xy(1-x-y)$ for extreme values	07
		OR	
Q.04	a	Evaluate (i) $\lim\limits_{x\to 0}\left(\frac{tanx}{x}\right)^{\frac{1}{x^2}}$ (ii) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$	06
	b	If $z = e^{ax+by}f(ax-by)$, show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$	07
	c	If $x+y+z = u,\ y+z = uv$ *and* $z = uvw$, *find* $\frac{\partial(x,y,z)}{\partial(u,v,w)}$	07

		Module-3	
Q. 05	a	Solve $x\frac{dy}{dx} + y = x^3y^6$	06
	b	Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ *is a parameter*	07
	c	Solve $xyp^2 - (x^2 + y^2)p + xy = 0$	07
		OR	
Q. 06	a	Solve $(x^2 + y^2 + x)dx + xydy = 0$	06
	b	A copper ball originally at 80° C cools down to 60° C in 20 minutes, if the temperature of the air being 40° C, what will be the temperature of the ball after 40 minutes from the original?	07
	c	Find the general solution of the equation $(px - y)(py + x) = a^2 p$ by reducing into Clairaut's form, taking the substitution $X = x^2$, $Y = y^2$	07
		Module-4	
Q. 07	a	Solve $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 2e^{3x} + 3$	06
	b	Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 1 + 3x + x^2$	07
	c	Using method of variation of parameters, solve $\frac{d^2y}{dx^2} + a^2y = \sec ax$	07
		OR	
Q. 08	a	Solve $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \cos 2x$	06
	b	Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin h(2x + 3)$	07
	c	Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1 + x)^2$	07
		Module-5	
Q. 09	a	Find the rank of the matrix $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix}$	06

	b	Solve the system of equations by using the Gauss-Jordan method $x+y+z=10,$ $2x-y+3z=19,$ $x+2y+3z=22$	07
	c	Using Rayleigh's power method find the dominant eigenvalue and the corresponding eigenvector of $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ by taking $[1 \quad 0 \quad 0]^T$ as initial eigen vector [carry out 6 iterations]	07
		OR	
Q. 10	a	Find the rank of the matrix $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$	06
	b	For what values λ and μ the system of equations $x+y+z=6; \quad x+2y+3z=10; \quad x+2y+\lambda z=\mu,$ has (i) no solution (ii) a unique solution and (iii) infinite number of solutions	07
	c	Solve the system of equations $2x-3y+20z=25;$ $20x+y-2z=17;$ $3x+20y-z=-18,$ Using the Gauss-Seidel method, taking (0, 0, 0) as an initial approximate root (Carry out 4 iterations).	07

Table showing the Bloom's Taxonomy Level, Course Outcome and Program Outcome

Question		Bloom's Taxonomy Level attached	Course Outcome	Program Outcome
Q.1	(a)	L1	CO 01	PO 01
	(b)	L2	CO 01	PO 01
	(c)	L3	CO 01	PO 02
Q.2	(a)	L1	CO 01	PO 01
	(b)	L2	CO 01	PO 01
	(c)	L3	CO 01	PO 02
Q.3	(a)	L2	CO 02	PO 01
	(b)	L2	CO 02	PO 01
	(c)	L3	CO 02	PO 03

Q.4	(a)	L2	CO 02	PO 01
	(b)	L2	CO 02	PO 01
	(c)	L3	CO 02	PO 02
Q.5	(a)	L2	CO 03	PO 02
	(b)	L3	CO 03	PO 03
	(c)	L2	CO 03	PO 01
Q.6	(a)	L2	CO 03	PO 02
	(b)	L3	CO 03	PO 03
	(c)	L2	CO 03	PO 01
Q.7	(a)	L2	CO 04	PO 01
	(b)	L2	CO 04	PO 01
	(c)	L2	CO 04	PO 02
Q.8	(a)	L2	CO 04	PO 01
	(b)	L2	CO 04	PO 01
	(c)	L2	CO 04	PO 02
Q.9	(a)	L2	CO 05	PO 01
	(b)	L3	CO 05	PO 01
	(c)	L3	CO 05	PO 02
Q.10	(a)	L2	CO 05	PO 01
	(b)	L3	CO 05	PO 02
	(c)	L3	CO 05	PO 01

Bloom's Taxonomy Levels	**Lower order thinking skills**		
	Remembering (Knowledge): L_1	Understanding (Comprehension): L_2	Applying (Application): L_3
	Higher-order thinking skills		
	Analyzing (Analysis): L_4	Valuating (Evaluation): L_5	Creating (Synthesis): L_6

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	Solution and Scheme for award of marks	

AY: 2021-22

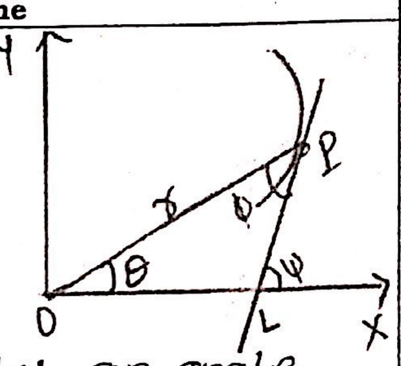
Department: Mathematics | **Model QP: 1**

Semester / Division: I / Common to All | **Name of Faculty: Prof. Umesh Poojari**

Subject with Sub. Code: Calculus & Differential Equations (21MAT11)

Q.No.	Solution and Scheme	Marks

1a) Let $P(r,\theta)$ be any point on a curve $r=f(\theta)$. Let $OP=r$. $\angle XOP=\theta$.



Let PL be the tangent to the curve at P which makes an angle ψ with +ve x-axis → (2)M

Also $\angle OPL=\phi$.

From figure, $\psi=\theta+\phi$

$$\tan\psi=\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} \quad —(1)$$

Let (x,y) be the cartesian co-ordinates of P $\therefore x=r\cos\theta,\ y=r\sin\theta$ → (2)M

Also, slope of the tangent is $\tan\psi=\frac{dy}{dx}$

$$\tan\psi=\frac{dy/d\theta}{dx/d\theta}=\frac{r\cos\theta+\frac{dr}{d\theta}\sin\theta}{-r\sin\theta+\frac{dr}{d\theta}\cos\theta}$$

divide both numerator and denominator by $\frac{dr}{d\theta}\cos\theta$

$$\tan\psi=\frac{r\frac{d\theta}{dr}+\tan\theta}{-r\frac{d\theta}{dr}\tan\theta+1} \quad —(2)$$ → (2)M

Comparing Equation (1) and (2)

$$\boxed{\tan\phi=r\frac{d\theta}{dr}}$$

1.b) $r = a(1+\cos\theta)$ — (i) and $r = b(1-\cos\theta)$

Take log on both sides and diff. wrt θ

$\log r = \log a + \log(1+\cos\theta)$, $\log r = \log b + \log(1-\cos\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{\sin\theta}{1+\cos\theta}$, $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$

$\cot\phi_1 = -\frac{2\sin\theta/2 \cdot \cos\theta/2}{2\cos^2\theta/2}$, $\cot\phi_2 = \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2}$

$= -\tan\theta/2 = \cot(\pi/2 + \theta/2)$, $= \cot\theta/2$

$\boxed{\phi_1 = \pi/2 + \theta/2}$ $\boxed{\phi_2 = \theta/2}$ ⟶ 3 + 3 M

∴ Angle of intersection is $= |\phi_1 - \phi_2| = \frac{\pi}{2}$. ⟶ 1 M

1.c) $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$

$\frac{dx}{d\theta} = a(1+\cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$

Now $\frac{dy}{dx} = y_1 = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2 \cos\theta/2}{2\cos^2\theta/2}$

$y_1 = \tan\theta/2$ ⟶ 2 M

Now $\frac{d^2y}{dx^2} = y_2 = \frac{1}{2}\sec^2\theta/2 \cdot \frac{d\theta}{dx}$

$= \frac{1}{2}\sec^2\theta/2 \left(\frac{1}{a(1+\cos\theta)}\right) = \frac{1}{4a}\sec^4(\theta/2)$ ⟶ 2 M

Radius of curvature

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2\theta/2)^{3/2}}{\frac{1}{4a}\sec^4(\theta/2)}$ ⟶ 2 M

$= 4a\,\frac{\sec^3\theta/2}{\sec^4\theta/2}$ ⟶ 1 M

$\boxed{\rho = 4a\cos\theta/2}$

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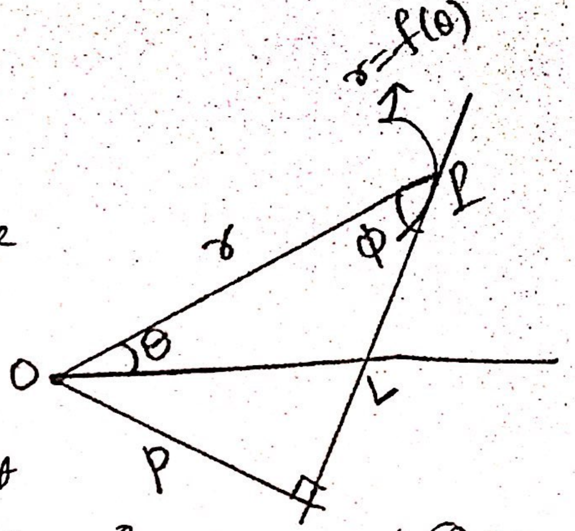
2 a) Let $P(r,\theta)$ be any point on $r=f(\theta)$. Let $OP=r$

Also ϕ be the angle made by the radius vector with the tangent.



Draw $ON \perp$ to the tangent

from the figure, $\angle ONP = 90°$ → ③ M

$\therefore \sin\phi = \frac{ON}{OP} = \frac{p}{r} \Rightarrow p = r\sin\phi$

Squaring and taking reciprocal → ② M

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1+\cot^2\phi)$$

$$= \frac{1}{r^2}\left(1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right) \qquad \left[\because \cot\phi = \frac{1}{r}\frac{dr}{d\theta}\right.$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$ → ① M

2 b). $r^m = a^m(\cos m\theta + \sin m\theta)$

$\log r^m = \log a^m + \log(\cos m\theta + \sin m\theta)$ → ① M

Diff. w.r.t θ

$$\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{-m\sin m\theta + m\cos m\theta}{\cos m\theta + \sin m\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos m\theta(1-\tan m\theta)}{\cos m\theta(1+\tan m\theta)} = \cot(\pi/4 + m\theta)$$ → ③ M

$\therefore \phi = \pi/4 + m\theta$.

Pedal Equation is $p = r\sin\phi = r\sin(\pi/4 + m\theta)$

$p = r(\sin\pi/4 \cdot \cos m\theta + \cos\pi/4 \cdot \sin m\theta)$ → ② M

$= \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta) = \frac{r}{\sqrt{2}}\frac{r^m}{a^m}$

$$\boxed{p = \frac{r^{m+1}}{\sqrt{2}\,a^m}}$$ → ① M

2. c) $x^3+y^3=3axy$

diff. w.r.t x, $3x^2+3y^2y_1=3axy_1+3ay$

$$y_1=\frac{ay-x^2}{y^2-ax} \quad — (1) \longrightarrow (2)M$$

at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $y_1=\dfrac{3a^2/2-9a^2/4}{9a^2/4-3a^2/2}=-1$

now diff (1) wrt x

$$y_2=\frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2} \longrightarrow (2)M$$

at $(3a/2, 3a/2)$

$$y_2=\frac{\left(9a^2/4-3a^2/2\right)(-a-3a)-\left(\frac{3a^2}{2}-\frac{9a^2}{4}\right)(-3a-a)}{\left(3a^2/4\right)^2} \longrightarrow (1)M$$

$$y_2=\frac{-32}{3a}$$

$\therefore$ Radius of curvature $\rho=\dfrac{(1+y_1^2)^{3/2}}{y_2} \longrightarrow (2)M$

$$\rho=\frac{(1+1)^{3/2}}{-32/3a}=\frac{2^{3/2}}{-32}\times 3a=\frac{-3a}{8\sqrt{2}}$$

$$\boxed{|\rho|=\left|\frac{-3a}{8\sqrt{2}}\right|}$$

Module – 2

3a) Let $y(x)=e^{\sin x}$ $\Rightarrow y(0)=e^{\sin 0}=1$ — (1)M

diff. wrt x

$y_1(x)=e^{\sin x}\cos x$ $\Rightarrow y_1(0)=e^{\sin 0}\cdot\cos 0=1$ $\longrightarrow (1)M$

$=y(x)\cdot\cos x$

$y_2(x)=y_1(x)\cos x-y(x)\sin x \Rightarrow y_2(0)=y_1(0)\cos 0-0=1$

$y_3(x)=y_2(x)\cos x-y_1(x)\sin x-y_1(x)\sin x-y(x)\cos x$

$=y_2(x)\cos x-2y_1(x)\sin x-y(x)\cos x$ $\longrightarrow (1)M$

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$\therefore y_3(0) = y_2(0)\cos 0 - 2y_1(0)\sin 0 - y(0)\cos 0$

$= 1 - 0 - 1 = 0 \longrightarrow 1M$

Also,

$y_4(x) = y_3(x)\cos x - 3y_2(x)\sin x - 3y_1(x)\cos x + y(x)\sin x.$

$y_4(0) = y_3(0)\cos 0 - 3y_2(0)\sin 0 - 3y_1(0)\cos 0 + y(0)\sin 0 \longrightarrow 1M$

$= -3$

Maclaurin's series is

$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0)$

$y(x) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}. \longrightarrow 1M$

3.b) $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

Let $u = f(p,q,r)$ where $\longrightarrow 1M$

$p = 2x-3y,\ q = 3y-4z,\ r = 4z-2x$

Now, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = 2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r}$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = -3\frac{\partial u}{\partial p} + 3\frac{\partial u}{\partial q} \rightarrow 3M$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} = -4\frac{\partial u}{\partial q} + 4\frac{\partial u}{\partial r}$

Now consider

$6u_x + 4u_y + 3u_z = 6\left(2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r}\right) + 4\left(-3\frac{\partial u}{\partial p} + 3\frac{\partial u}{\partial q}\right) + 3\left(-4\frac{\partial u}{\partial q} + 4\frac{\partial u}{\partial r}\right)$

$= 12\frac{\partial u}{\partial p} - 12\frac{\partial u}{\partial r} - 12\frac{\partial u}{\partial p} + 12\frac{\partial u}{\partial q} + 12\frac{\partial u}{\partial q} + 12\frac{\partial u}{\partial r}$

$\longrightarrow 3M$

$\therefore 6u_x + 4u_y + 3u_z = 0.$

3. c) $f(x,y) = xy(1-x-y) = xy - x^2y - xy^2$

$f_x = y - 2xy - y^2, \quad f_y = x - x^2 - 2xy$ → (1) M

$A = f_{xx} = -2y, \quad C = f_{yy} = -2x, \quad B = f_{xy} = 1-2x-2y$ → (1) M

Now $f_x = 0$ and $f_y = 0$

$y(1-2x-y) = 0, \quad x(1-x-2y) = 0$

$y = 0,\ 2x+y = 1, \quad x = 0,\ x+2y = 1$ → (1) M

∴ stationary points are $(0,0)\ (1,0),\ (0,1)$ and $(1/3, 1/3)$.

	$(0,0)$	$(1,0)$	$(0,1)$	$(1/3,1/3)$
$A=-2y$	0	0	-2	$-2/3$
$B=1-2x-2y$	1	-1	-1	$-1/3$
$C=-2x$	0	-2	0	$-2/3$
$AC-B^2$	$-1<0$	$-1<0$	$-1<0$	$1/3>0$
Conclusion	Saddle pt	Saddle pt	Saddle pt	Max.

→ (2) M

$f(x,y)$ is maximum at $(1/2, 1/3)$ and the maximum value is

$$f(1/3, 1/3) = 1/3 \cdot 1/3 (1 - 1/3 - 1/3) = \frac{1}{27}.$$ → (1) M

4 a) (i) $k = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$, 1^∞ form

$$\log k = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2}, \ \frac{0}{0} \text{ form}$$ → (1) M

By L'Hospital's rule

$$\log k = \lim_{x\to 0} \left[\frac{\frac{x}{\tan x}\left[\frac{x\sec^2 x - \tan x}{x^2}\right]}{2x}\right]$$

$$= \lim_{x\to 0} \frac{x}{\tan x} \cdot \lim_{x\to 0} \left(\frac{x\sec^2 x - \tan x}{2x^3}\right)$$ → (1) M

$$= 1 \cdot \lim_{x\to 0} \frac{\sec^2 x \tan x}{3x}$$

$$= \lim_{x\to 0} \frac{\tan x}{x} \cdot \lim_{x\to 0} \frac{\sec^2 x}{3} \rightarrow (1)M$$

$$= \frac{\sec^2 0}{3} = 1/3 \Rightarrow \boxed{k = e^{1/3}}$$

(ii) $k = \lim_{x\to 0} (\cos x)^{1/x^2}$, 1^∞ form

$\therefore \log k = \lim_{x\to 0} \frac{1}{x^2} \log(\cos x)$, $\infty \times 0$ form $\rightarrow$ (1)M

$$= \lim_{x\to 0} \frac{\log \cos x}{x^2}, \ \frac{0}{0} \text{ form}$$

By L'Hospital's rule $\rightarrow$ (1)M

$$= \lim_{x\to 0} \frac{\left(\frac{\sin x}{\cos x}\right)}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\tan x}{x} \rightarrow (1)M$$

$$\log k = \frac{1}{2} \cdot 1 \Rightarrow \boxed{k = e^{1/2}}$$

4 b) $z = e^{ax+by} f(ax-by)$

$\frac{\partial z}{\partial x} = a e^{ax+by} f(ax-by) + a\, e^{ax+by} f'(ax-by) \rightarrow$ (2)M

$= az + a e^{ax+by} f'(ax-by)$

$\frac{\partial z}{\partial y} = b e^{ax+by} f(ax-by) - b\, e^{ax+by} f'(ax-by) \rightarrow$ (2)M

$= bz - b e^{ax+by} f'(ax-by)$

Now consider

$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = b\left(az + a e^{ax+by} f'(ax-by)\right) \rightarrow$ (2)M

$+ a\left(bz - b e^{ax+by} f'(ax-by)\right)$ $\rightarrow$ (1)M

$= 2abz$

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4.c) $x+y+z=u$ — ①, $y+z=uv$ — ②

$z=uvw$ — ③

Here we need to express x,y,z in terms of u,v,w. → ② M

$x=u-(y+z)=u-uv$

$y=uv-z=uv-uvw$

$z=uvw$.

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix} \rightarrow ① M$$

$$=\begin{vmatrix}1-v & -u & 0\\ v-vw & u-uw & -uv\\ vw & uw & uv\end{vmatrix}=u^2v(1-w) \rightarrow ④ M$$

∴ Module-3

5 a). $x\frac{dy}{dx}+y=x^3y^6$

Dividing by xy^6

$\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{xy^5}=x^2$, put $\frac{1}{y^5}=t$ → ② M

$-5\frac{1}{y^6}\frac{dy}{dx}=\frac{dt}{dx}$

$\therefore -\frac{1}{5}\frac{dt}{dx}+\frac{t}{x}=x^2$

$\frac{dt}{dx}-\frac{5}{x}t=-5x^2$

a linear D.E. in t, $P=-5/x$, $Q=-5x^2$

now $e^{\int P dx}=e^{-\int 5/x\, dx}=e^{-5\log x}=\frac{1}{x^5}$ → ② M

∴ Solution is $t\, e^{\int P dx}=\int Q\, e^{\int P dx}dx+c$

$t(1/x^5)=\int -5x^2(1/x^5)\,dx+c$ → ② M

$$\boxed{\frac{1}{x^5y^5}=\frac{5}{2x^2}+c}$$

5.b) $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ —— ①

Diff. w.r.t $x \Rightarrow \frac{2x}{a^2} + \frac{2y\, dy/dx}{b^2+\lambda} = 0$ → ① M

$$\frac{y}{b^2+\lambda} = \frac{-x}{a^2(dy/dx)}$$

Equation is $\frac{x^2}{a^2} - \frac{xy}{a^2(dy/dx)} = 1$ → ② M

$$x^2\frac{dy}{dx} - xy = a^2\frac{dy}{dx}$$

$$(x^2-a^2)\frac{dy}{dx} = xy$$

Now replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ → ① M

$$(x^2-a^2)\left(-\frac{dx}{dy}\right) = xy$$

Integrating $\int \frac{a^2-x^2}{x}\,dx = \int y\,dy + c$ → ② M

$$a^2\log x - \frac{x^2}{2} = \frac{y^2}{2} + c$$

$\boxed{2a^2\log x - x^2 - y^2 = 2c}$ is the → ① M

required orthogonal trajectories

5c) $xyp^2 - (x^2+y^2)p + xy = 0$

$$p = \frac{(x^2+y^2) \pm \sqrt{(x^2+y^2)^2 - 4x^2y^2}}{2xy}$$

$$= \frac{x^2+y^2 \pm (x^2-y^2)}{2xy}$$ → ③ M

$p = \frac{x}{y}$ or $p = y/x$

$\frac{dy}{dx} = x/y$ or $\frac{dy}{dx} = y/x$ → ② M

$y\,dy = x\,dx$ or $\frac{dy}{y} = \frac{dx}{x}$

Integrating $y^2 - x^2 - c = 0$ or $\log y - \log x = \log c$

$(y^2-x^2-c)(y-cx) = 0$. → ② M

6. a) $(x^2+y^2+x)\,dx + xy\,dy = 0$

This is in the form $M\,dx + N\,dy = 0$

$M = x^2+y^2+x$, $N = xy$ → ②M

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Equation is not Exact

Now $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = y$ which is similar to N.

$\therefore f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{y}{xy} = 1/x$ → ②M

$e^{\int f(x)dx} = e^{\int 1/x\, dx} = e^{\log x} = x$

Now $M = x^3+xy^2+x^2$, $N = x^2y$

$\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = 2xy$ Equation is Exact

$\therefore$ Sol^n is $\int\limits_{y \to \text{const}} M\,dx + \int N(y)\,dy = c$ → ②M

$\int (x^3+xy^2+x^2)\,dx + 0 = c$

$\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$ is the solution

6. b) Here $t_1 = 80$, $t_2 = 40$, $T = 60$ at $t = 20$ → ①M

By Newton's law of cooling

$T = t_2 + (t_1 - t_2)\,e^{-kt}$ → ①M

$60 = 40 + 40\,e^{-20k} \Rightarrow k = \frac{1}{20}\log_e 2 = 0.0346$ → ②M

We have to find T when $t = 40$

$T = t_2 + (t_1 - t_2)\,e^{-kt}$ → ②M

$= 40 + 40\,e^{-(0.0346)40}$

$\boxed{T = 50^\circ C}$ → ①M

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6.c) $X = x^2$, $Y = y^2$

$\frac{dX}{dx} = 2x$, $\frac{dY}{dy} = 2y$ → ① M

Now $p = \frac{dy}{dx} = \frac{dy}{dY} \cdot \frac{dY}{dX} \frac{dX}{dx} = \frac{x}{y} \frac{dY}{dX} = \frac{x}{y} P$

$p = \frac{\sqrt{X}}{\sqrt{Y}} P$ → ② M

Now $(px-y)(py+x) = a^2 p$

$\left(\frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{Y} + \sqrt{X}\right) = a^2 \frac{\sqrt{X}}{\sqrt{Y}} P$

$\left(\frac{PX-Y}{\sqrt{Y}}\right)\sqrt{X}(P+1) = a^2 \frac{\sqrt{X}}{\sqrt{Y}} P$ → ③ M

$PX - Y = \frac{a^2 P}{P+1}$

$Y = XP - \frac{a^2 P}{P+1}$ is in the

Clairaut's form.

∴ g.s. is $Y = XC - \frac{a^2 C}{C+1}$ → ① M

OR $\boxed{y^2 = x^2 c - \frac{a^2 c}{c+1}}$

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Module-04

7.a) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 2e^{3x} + 3$

$(D^3 - 6D^2 + 11D - 6)y = 2e^{3x} + 3$

solution is $y = C.F + P.I = y_c + y_p$

C.F (y_c): Now $(D^3 - 6D^2 + 11D - 6)y = 0$ → ② M

A.E is $m^3 - 6m^2 + 11m - 6 = 0$

$m = 1, 2, 3$

$\therefore y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

P.I (y_p): Now $y_p = \frac{2e^{3x} + 3}{D^3 - 6D^2 + 11D - 6}$

$= \frac{2e^{3x}}{D^3 - 6D^2 + 11D - 6} + \frac{3}{D^3 - 6D^2 + 11D - 6}$ → ② + ① M

$= \frac{2x\,e^{3x}}{3D^2 - 12D + 11} + \frac{3}{-6}$

$= \frac{2x\,e^{3x}}{3(9) - 12(3) + 11} - \frac{1}{2} = \frac{2x e^{3x}}{2} - \frac{1}{2}$

$= xe^{3x} - \frac{1}{2}$ → ① M

$\therefore y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} + xe^{3x} - \frac{1}{2}$

7.b) $(D^2 - 2D + 1)y = 1 + 3x + x^2$

$y = y_c + y_p = C.F + P.I$

y_c: Now $(D^2 - 2D + 1)y = 0$ → ③ M

A.E. is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$y_c = (c_1 + c_2 x)e^{x}$

y_p: $y_p = \frac{x^2 + 3x + 1}{1 - 2D + D^2} \Rightarrow$ → ③ M

$$\begin{array}{r|l} & x^2 + 7x + 13 \\ \hline 1 - 2D + D^2 & x^2 + 3x + 1 \\ & -x^2 \mp 4x + 2 \\ \hline & 7x - 1 \\ & 7x - 14 \\ \hline & 13 \\ & -13 \\ \hline & 0 \end{array}$$

$\therefore \boxed{y = c_1 e^{x} + c_2 x e^{x} + x^2 + 7x + 13}$ → ① M

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7. c) $\frac{d^2y}{dx^2} + a^2y = \sec ax$

$(D^2+a^2)y = \sec ax$ → (2) M

$m^2+a^2=0 \Rightarrow m = \pm ai$

$\therefore y_c = C_1\cos ax + C_2\sin ax$

Let $y = Ay_1 + By_2$ be the complete solution

$y_1 = \cos ax, \quad y_2 = \sin ax$ → (2) M

$W = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = a$

$A = -\int \frac{y_2\phi(x)}{W}dx = -\int \frac{\sin ax \cdot \sec ax}{a}dx + k_1$

$= -\frac{1}{a}\int \tan ax\, dx + k_1 = -\frac{\log \sec ax}{a^2} + k_1$ → (3) M

$B = \int \frac{y_1\phi(x)}{W}dx + k_2 = \int \frac{\cos ax \cdot \sec ax}{a}dx + k_2 = \frac{x}{a} + k_2$

$\therefore$ G.S. is $y = k_1\cos ax + k_2\sin ax - \frac{\cos ax \cdot \log \sec ax}{a^2}$ → (1) M

$+ \frac{x\sin ax}{a}$

8 a) $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \cos 2x \Rightarrow (D^3+4D)y = \cos 2x$

A.E. is $m^3+4m=0 \Rightarrow m = 0, \pm 2i$ → (2) M

$y_c = C_1 + C_2\cos 2x + C_3\sin 2x$

$y_p = \frac{\cos 2x}{D^3+4D}$ Replace D^2 by $-a^2 = -4$ → (2) M

$= \frac{\cos 2x}{-4D+4D}$ Here Denominator is zero

$= x\left(\frac{\cos 2x}{3D^2+4}\right) = \frac{x\cos 2x}{3(-4)+4} = \frac{x\cos 2x}{-8}$ → (2) M

$\therefore y = y_c + y_p = C_1 + C_2\cos 2x + C_3\sin 2x$

$- \frac{x}{8}\cos 2x$

8.b) $(D^2+3D+2)y = \sinh(2x+3) = \frac{1}{2}\left[e^{2x+3} - e^{-(2x+3)}\right]$

A.E. is $m^2+3m+2=0 \Rightarrow m=-1,-2$ $\rightarrow$ 2 M

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$$y_p = \frac{1}{2}\left[\frac{e^{2x+3}}{D^2+3D+2} - \frac{e^{-(2x+3)}}{D^2+3D+2}\right] \rightarrow \text{2 M}$$

$$= \frac{1}{2}\left[\frac{e^{2x+3}}{4+6+2} - \frac{e^{-(2x+3)}}{2D+3}\right] \rightarrow \text{2 M}$$

$$= \frac{1}{2}\left[\frac{e^{2x+3}}{12} + \frac{e^{-(2x+3)}}{1}\right]$$

$$\therefore y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2}\left[\frac{e^{2x+3}}{12} + e^{-(2x+3)}\right] \rightarrow \text{1 M}$$

8.c) $x^2 y'' - 3xy' + 4y = (1+x)^2$ —— (1) $\rightarrow$ 1 M

put $t = \log x$ (or) $x = e^t$

$xy' = Dy$, $x^2y'' = D(D-1)y$

$\therefore$ Eq^n (1) $\Rightarrow$ $(D^2-4D+4)y = 1+2e^t+e^{2t}$ $\rightarrow$ 2 M

A.E. is $m^2-4m+4=0 \Rightarrow m=2,2$ $\rightarrow$ 1 M

$y_c = (c_1+c_2 t)\, e^{2t}$

$$y_p = \frac{1}{D^2-4D+4} + \frac{2e^t}{D^2-4D+4} + \frac{e^{2t}}{D^2-4D+4} \rightarrow \text{2 M}$$

$$= \frac{1}{4} + \frac{2e^t}{1-4+4} + \frac{e^{2t}}{4-8+4}$$

$$= \frac{1}{4} + 2e^t + \frac{e^{2t}}{2D-4}\cdot t$$

$$= \frac{1}{4} + 2e^t + \frac{t^2 e^{2t}}{2}$$

$\therefore y = y_c + y_p$

$$= (c_1 + c_2 \log x)x^2 + \frac{1}{4} + 2x + \frac{x^2(\log x)^2}{2} \rightarrow \text{1 M}$$

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9a) $A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 4 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix}$

$R_1 \leftrightarrow R_2 \Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & -1 & 3 \\ 3 & 6 & 12 & 9 \\ 3 & 3 & 3 & 6 \end{bmatrix}$ → ② M

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 3R_1$
$A \sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -3 & -9 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -9 & -3 \end{bmatrix}$ → ② M

$R_4 \to R_4 - R_2$
$A \sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -3 & -9 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → ② M

Rank $= \rho(A) = 2$

9b) $x+y+z=10$, $2x-y+3z=19$ $x+2y+3z=22$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 10 \\ 19 \\ 22 \end{bmatrix}$

$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 19 \\ 1 & 2 & 3 & 22 \end{array}\right]$ → ① M

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$\Rightarrow [A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 2 & 12 \end{array}\right]$ → ② M

$R_3 \to 3R_3 + R_2 \Rightarrow [A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 7 & 35 \end{array}\right]$

→ ② M

$R_1 \to 3R_1 + R_2$
$[A:B] \sim \left[\begin{array}{ccc:c} 3 & 0 & 4 & 29 \\ 0 & -3 & 1 & -01 \\ 0 & 0 & 7 & 35 \end{array}\right]$

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$R_1 \to 7R_1 - 4R_3$
$R_2 \to 7R_2 - R_3$
$\Rightarrow [A:B] \sim \begin{bmatrix} 21 & 0 & 0 & : 63 \\ 0 & -21 & 0 & : -42 \\ 0 & 0 & 7 & : 35 \end{bmatrix} \to$ ② M

$\Rightarrow 21x = 63$
$-21y = -42 \quad \Rightarrow x = 3,\ y = 2,\ z = 5$
$7z = 35$

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9 c). $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & \cancel{7} & 5 \end{bmatrix},\ X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$AX^{(0)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)} \to$ ① M

$AX^{(1)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ +0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \lambda^{(2)} X^{(2)} \to$ ① M

$AX^{(2)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6\begin{bmatrix} 1 \\ 0.928 \\ -0.928 \end{bmatrix} = \lambda^{(3)} X^{(3)} \to$ ① M

$AX^{(3)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.928 \\ -0.928 \end{bmatrix} = \begin{bmatrix} 5.857 \\ 5.714 \\ -5.714 \end{bmatrix} = 5.857\begin{bmatrix} 1 \\ 0.97 \\ -0.97 \end{bmatrix} \to$ ① M
$= \lambda^{(4)} X^{(4)}$.

$AX^{(4)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.97 \\ -0.97 \end{bmatrix} = \begin{bmatrix} 5.95 \\ 5.90 \\ -5.90 \end{bmatrix} = 5.95\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} \Rightarrow \lambda^{(5)} X^{(5)} \to$ ① M

$AX^{(5)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.93 \\ -5.93 \end{bmatrix} = 5.98\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} \to$ ① M

$\therefore$ largest Eigen value is 5.98
and Eigen vector is $[1 \ \ 0.99 \ \ -0.99]^T$

$\longrightarrow$ ① M

10 a)

$$A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$ → ② M

$$A \sim \begin{bmatrix} 11 & 12 & 13 & 14 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 11 & 12 & 13 & 14 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 11R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_3 - 3R_1 \end{matrix}$$ → ② M

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → ② M

Rank $= \rho(A) = 02$

10 b)

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$ $\Rightarrow$ $[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\Rightarrow$ $[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$ → ② M

a) No solution:

We must have $\rho[A] \neq \rho[A:B]$

If $\lambda = 3$ then $\rho[A] = 2$ and if $\mu \neq 10$ → ① M

then $\rho[A:B] = 3$

b) Unique solution:

We must have $\rho[A] = \rho[A:B] = 3$

$\rho[A] = 3$ if $\lambda \neq 3$ and $\rho[A:B] = 3$ → ② M

for any value of λ.

$\therefore \lambda \neq 3$. for unique solution

c) Infinite number of solutions:

Here we must have $\rho[A] = \rho[A:B] < 3$

$\rho[A] = \rho[A:B] = 2$ only if $\lambda = 3$ and $\mu = 10$.

$\therefore$ System will have infinite solution → ② M

if $\lambda = 3$, $\mu = 10$

10 c)

$$2x - 3y + 20z = 25$$
$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$

Here system is not a diagonally dominant, rearranging the equations

$20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$ → ① M

$x = \frac{1}{20}[17 - y + 2z]$

$y = \frac{1}{20}[-18 - 3x + z]$, $z = \frac{1}{20}[25 - 2x + 3y]$ → ① M

Let $x = 0$, $y = 0$, $z = 0$

1st iteration: $x^{(1)} = \frac{17}{20} = 0.85$ → ① M

$y^{(1)} = \frac{1}{20}[-18 - 3(0.85)] = -1.0275$

$z^{(1)} = \frac{1}{20}[25 - 2(0.85) + 3(-1.0275)] = 1.0109$

2nd iteration:

$x^{(2)} = \frac{1}{20}[17 - (-1.0275) + 2(1.0109)] = 1.0025$

$y^{(2)} = \frac{1}{20}[-18 - 3(1.0025) + 1.0109] = -0.9998$ → ① M

$z^{(2)} = \frac{1}{20}[25 - 2(1.0025) + 3(-0.9998)] = 0.9998$

3rd iteration;

$x^{(3)} = \frac{1}{20}[17-(-0.9998)+2(0.9998)] = 0.9999$

$y^{(3)} = \frac{1}{20}[-18-3(0.9999)+0.9999] = -1.000005$ $\rightarrow$ 1M

$z^{(3)} = \frac{1}{20}[25-2(0.99997)+3(-1.0000005)] = 1.0000002$

4th iteration;

$x^{(4)} = \frac{1}{20}[17-(0.9999)+2(1.0000002)] = 1$ $\rightarrow$ 1M

$y^{(4)} = \frac{1}{20}[-18-3(1)+1.0000002] = -1$

$z^{(4)} = \frac{1}{20}[25-2(1)+3(-1)] = 1$

$\rightarrow$ 1M.

$\therefore x = 1 \quad y = -1 \quad z = 1$

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