Model Question Paper-II with effect from 2021 (CBCS Scheme)

USN

First Semester B.E Degree Examination
Calculus and Differential Equations (21MAT11)

TIME: 03 Hours **Max. Marks: 100**

Note: Answer any **FIVE** full questions, choosing at least **ONE** question from each **MODULE.**

		Module -1	**Marks**
Q.01	a	With usual notations prove that $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$	**06**
	b	Find the angle between the curves $r = a\, log\theta$ and $r = \frac{a}{log\theta}$	**07**
	c	Prove that for the cardioids $r = a(1+\cos\theta)$, $\frac{\rho^2}{r}$ is constant	**07**
		OR	
Q.02	a	Show that the curves $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut each other orthogonally.	**06**
	b	Find the pedal equation of the curve $r^n = a^n \cos n\theta$.	**07**
	c	Show that the radius of curvature at $(a, 0)$ on the curve $y^2 = \frac{a^2(a-x)}{x}$ is $\frac{a}{2}$	**07**
		Module-2	
Q. 03	a	Expand $\sqrt{1+sin2x}$ by Maclaurin's series up to the term containing x^5	**06**
	b	If $u = tan^{-1}\left(\frac{y}{x}\right)$ $, where\ x = e^t - e^{-t} and\ y = e^t + e^{-t}$, find the total derivative $\frac{du}{dt}$ using partial differentiation.	**07**
	c	If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$	**07**
		OR	
Q.04	a	Evaluate (i) $\lim_{x\to 0} (a^x + x)^{\frac{1}{x}}$ (ii) $\lim_{x\to\frac{\pi}{2}} (\tan x)^{\tan 2x}$	**06**
	b	If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$	**07**
	c	Find the extreme values of $x^3 + y^3 - 3axy$, a ≥ 0	**07**
		Module-3	
Q. 05	a	Solve $\frac{dy}{dx} + y\, tan\, x = y^3 sec\, x$	**06**
	b	Water at temperature $10°C$ takes 5 minutes to warm up to $20°C$ at a room temperature of $40°C$. Find the temperature of the water after 20 minutes.	**07**

	c	Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$	07
OR			
Q. 06	a	Solve $(x^2 + y^3 + 6x)dx + y^2xdy = 0$	06
	b	Prove that the system of parabolas $y^2 = 4a(x + a)$ are self-orthogonal	07
	c	Find the general and singular solutions of $xp^2 + xp - yp + 1 - y = 0$	07
Module-4			
Q. 07	a	Solve $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$	06
	b	Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$	07
	c	Using the method of Variation of parameters, solve $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$$	07
OR			
Q. 08	a	Solve $(\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y) = e^{2x}$	06
	b	Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = sinx$	07
	c	Solve $(1 + x)^2\frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + y = \sin[2\log(1 + x)]$	07
Module-5			
Q. 09	a	Find the rank of the matrix $$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$	06
	b	Solve the system of equations by using the Gauss elimination method $3x + y + 2z = 3,$ $2x - 3y - z = -3,$ $x + 2y + z = 4$	07
	c	Using the Gauss-Seidel iteration method, solve the equations $83x + 11y - 4z = 95;$ $3x + 8y + 29z = 71;$ $7x + 52y + 13z = 104,$ Carry out four iterations, starting with the initial approximations (0, 0, 0)	07
OR			
Q. 10	a	Test for consistency and solve $5x + 3y + 7z = 4\ ; 3x + 26y + 2z = 9\ ; 7x + 2y + 10z = 5$	06
	b	Using the Gauss Jordan method, solve	07

		$x + y + z = 11;\ 3x - y + 2z = 12;\ 2x + y - z = 3$	
	c	Find the largest eigenvalue and the corresponding eigenvector of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ with the initial approximate eigenvector $[1 \quad 0 \quad 0]^T$	07

Table showing the Bloom's Taxonomy Level, Course Outcome and Program Outcome

Question		Bloom's Taxonomy Level attached	Course Outcome	Program Outcome
Q.1	(a)	L1	CO 01	PO 01
	(b)	L2	CO 01	PO 01
	(c)	L3	CO 01	PO 02
Q.2	(a)	L1	CO 01	PO 01
	(b)	L2	CO 01	PO 01
	(c)	L3	CO 01	PO 02
Q.3	(a)	L2	CO 02	PO 01
	(b)	L2	CO 02	PO 01
	(c)	L3	CO 02	PO 02
Q.4	(a)	L2	CO 02	PO 01
	(b)	L2	CO 02	PO 01
	(c)	L3	CO 02	PO 03
Q.5	(a)	L2	CO 03	PO 02
	(b)	L3	CO 03	PO 03
	(c)	L2	CO 03	PO 01
Q.6	(a)	L2	CO 03	PO 01
	(b)	L3	CO 03	PO 03
	(c)	L2	CO 03	PO 02
Q.7	(a)	L2	CO 04	PO 01

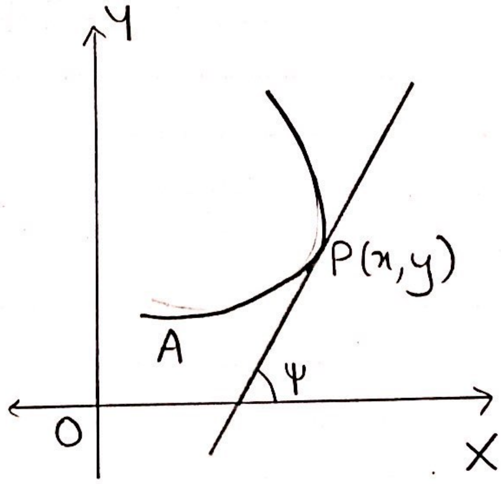
	(b)	L2	CO 04	PO 02
	(c)	L2	CO 04	PO 03
Q.8	(a)	L2	CO 04	PO 01
	(b)	L2	CO 04	PO 02
	(c)	L2	CO 04	PO 03
Q.9	(a)	L2	CO 05	PO 02
	(b)	L3	CO 05	PO 01
	(c)	L3	CO 05	PO 01
Q.10	(a)	L2	CO 05	PO 02
	(b)	L3	CO 05	PO 01
	(c)	L3	CO 05	PO 01

Bloom's Taxonomy Levels	**Lower order thinking skills**		
	Remembering (Knowledge): L_1	Understanding (Comprehension): L_2	Applying (Application): L_3
	Higher-order thinking skills		
	Analyzing (Analysis): L_4	Valuating (Evaluation): L_5	Creating (Synthesis): L_6

AY: 2021-22

Department: Mathematics | **Model QP: 2**
Semester / Division: I / Common to All | **Name of Faculty: Dr. Meenal M. Kaliwal**
Subject with Sub. Code: Calculus & Differential Equations (21MAT11)

Q.No.	Solution and Scheme	Marks
1 a.	With usual notations prove that $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$	
Sol^n	Let $y = f(x)$ be the equation of the cartesian curve and A be a fixed point on it. Let, $P(x, y)$ be a point on the curve such that $\widehat{AP} = s$. Let ψ be the angle made by the tangent at P with the x-axis.	1M
	We know that, $\tan\psi = dy/dx$	
	Differentiating with respect to 's',	
	$\dfrac{d}{ds}(\tan\psi) = \dfrac{d}{ds}\left(\dfrac{dy}{dx}\right)$	1 M
	$\sec^2\psi \, \dfrac{d\psi}{ds} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)\dfrac{dx}{ds}$	
	But, $\dfrac{dx}{ds} = \cos\psi$	
	and by the definition	
	$\dfrac{d\psi}{ds} = \dfrac{1}{\rho}$	



$$\therefore \sec^2\psi \cdot \frac{1}{\rho} = \frac{d^2y}{dx^2}\cos\psi \quad \text{or}$$

$$\sec^3\psi = \rho\,\frac{d^2y}{dx^2}$$

Hence, $\rho = \sec^3\psi \Big/ \frac{d^2y}{dx^2}$ 1M

ie. $\rho = \frac{(\sec^2\psi)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{3/2}}{\frac{d^2y}{dx^2}}$ 1M

$$= \frac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2}$$ 1M

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$ 1M

where, $y_1 = dy/dx$ & $y_2 = d^2y/dx^2$

Total → 6M

1 b. Find the angle between the curves $r = a\log\theta$ and $r = a/\log\theta$.

Soln.

$$\log r = \log a + \log(\log\theta)$$

Differentiating with respect to 'θ',

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\log\theta \cdot \theta}$$

$$\cot\phi_1 = \frac{1}{\theta\log\theta} \longrightarrow (1)$$ 1M

Also, $r = \frac{a}{\log\theta}$

$$\log r = \log a - \log(\log\theta)$$

Differentiating wrt 'θ',

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$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$ 1M

$$= -\tan(\theta/2)$$

$$\therefore\ r_1 = -r\tan(\theta/2) \longrightarrow (1)$$ 1M

Again differentiating eqn (1) wrt 'θ',

$$r_2 = -\frac{r}{2}\sec^2\left(\frac{\theta}{2}\right) - r_1\tan(\theta/2)$$

$$r_2 = -\frac{r}{2}\sec^2\left(\frac{\theta}{2}\right) + r\tan(\theta/2)(\tan\theta/2)$$

$$r_2 = -\frac{r}{2}\sec^2\left(\frac{\theta}{2}\right) + r\tan^2\left(\frac{\theta}{2}\right)$$

We have, $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$ 1M

$$= \frac{\{r^2+r^2\tan^2(\theta/2)\}^{3/2}}{r^2+2r^2\tan^2(\theta/2)+\frac{r^2}{2}\sec^2(\theta/2) - r^2\tan^2(\theta/2)}$$

$$= \frac{(r^2)^{3/2}\{1+\tan^2(\theta/2)\}^{3/2}}{r^2+r^2\tan^2\left(\frac{\theta}{2}\right)+\frac{r^2}{2}\sec^2(\theta/2)}$$ 1M

$$= \frac{r^3\{\sec^2(\theta/2)\}^{3/2}}{r^2\left\{1+\tan^2\left(\frac{\theta}{2}\right)+\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\right\}}$$ 1M

$$= \frac{r\sec^3(\theta/2)}{\cancel{r^2}\ \frac{3}{2}\sec^2\left(\frac{\theta}{2}\right)} = \frac{2r}{3}\sec\left(\frac{\theta}{2}\right)$$

$$\therefore\ \rho = \frac{2r}{3}\sec(\theta/2) \longrightarrow (2)$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\log\theta . \theta}$$

$$\cot\phi_2 = -\frac{1}{\theta . \log\theta} \longrightarrow (2)$$ 1M

From (1) & (2) we have,

$\tan\phi_1 = \theta\log\theta$; $\tan\phi_2 = -\theta\log\theta$

Consider, $\tan(\phi_1-\phi_2) = \dfrac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1 . \tan\phi_2}$ 1M

ie. $\tan(\phi_1-\phi_2) = \dfrac{2\theta\log\theta}{1-(\theta\log\theta)^2} \longrightarrow (3)$ 1M

To find θ we solve the given pair of equations

$r = a\log\theta$ & $r = a/\log\theta$

Equating the RHS we have

$$a\log\theta = \frac{a}{\log\theta}$$

$(\log\theta)^2 = 1$ or $\log\theta = 1 \Rightarrow \theta = e$ 1M

Substituting $\theta = e$ in eqn (3), we get

$\tan(\phi_1-\phi_2) = \dfrac{2e}{1-e^2}$ $(\because \log_e e = 1)$ 1M

Thus, the angle of intersection $= \phi_1 - \phi_2$

$$= \tan^{-1}\left(\frac{2e}{1-e^2}\right)$$

$$= 2\tan^{-1} e$$ 1M

Total → 7M

1c. Prove that for the cardiods $r = a(1+\cos\theta)$, ρ^2/r is constant.

Sol^n Given, $r = a(1+\cos\theta)$

$$\log r = \log a + \log(1+\cos\theta)$$

Differentiating with respect to 'θ', But, $r = a(1+\cos\theta) = a \cdot 2\cos^2(\theta/2)$

$\therefore \sec^2(\theta/2) = \frac{2a}{r}$ or $\sec(\theta/2) = \frac{\sqrt{2a}}{\sqrt{r}}$

Hence, ② becomes

$$\rho = \frac{2r}{3} \times \frac{\sqrt{2a}}{\sqrt{r}} \qquad 1M$$

$$\rho = \frac{2}{3}\sqrt{2ar}$$

$\therefore \rho^2 = \frac{4}{9}(2ar)$ or $\frac{\rho^2}{r} = \frac{8a}{9}$ = constant 1M

Thus, ρ^2/r is a constant.

Total → 7M

OR

2a. Show that the curves $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut each other orthogonally.

Sol^n Given, $r^n = a^n \cos n\theta$; $r^n = b^n \sin n\theta$

Taking logarithms,

$$n \log r = n \log a + \log \cos n\theta$$

Differentiating wrt 'θ',

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{(-\sin n\theta) \times n}{\cos n\theta}$$

$$\frac{n}{r}\frac{dr}{d\theta} = -n\tan n\theta \qquad 1M$$

$$\cot\phi_1 = \cot\left(\frac{\pi}{2} + n\theta\right).$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} + n\theta \qquad 1M$$

Consider, $r^n = b^n \sin n\theta$

$$n \log r = n \log b + \log \sin n\theta$$

Differentiating with respect to 'θ',

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n\cos n\theta}{\sin n\theta}$$ 1M

$$\frac{1}{r}\frac{dr}{d\theta} = \cot n\theta \Rightarrow \cot\phi_2 = \cot n\theta$$ 1M

$$\Rightarrow \phi_2 = n\theta$$ 1M

$$\therefore |\phi_1 - \phi_2| = |\pi/2 + n\theta - n\theta| = \pi/2$$ 1M

Thus, the curves intersect each other orthogonally.

Total → 6M

2b. Find the pedal equation of the curve $r^n = a^n \cos n\theta$.

Sol^n. Given, $r^n = a^n \cos n\theta$

$$n \log r = n \log a + \log \cos n\theta$$

Differentiating wrt 'θ',

$$\frac{n}{r} \times \frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta} = -n\tan n\theta$$ 2M

$$\cot\phi = \cot(\pi/2 + n\theta)$$

$$\Rightarrow \phi = \pi/2 + n\theta$$ 1M

Consider, $p = r\sin\phi$ 1M

$$p = r\sin\left(\frac{\pi}{2} + n\theta\right) = r\cos n\theta$$

We have, $r^n = a^n \cos n\theta \longrightarrow (1)$

$p = r\cos n\theta \longrightarrow (2)$ 1M

$\therefore$ (1) as a consequence of (2) is

$r^n = a^n (p/r) \longrightarrow (3)$ 1M

Thus,

$\underline{r^{n+1} = pa^n}$ is the required pedal equation. 1M

Total → 7M

2c. Show that the radius of curvature at $(a,0)$ on the curve $y^2 = \frac{a^2(a-x)}{x}$ is $a/2$.

Sol^n Given curve is, $y^2 = \frac{a^2(a-x)}{x}$

$$y^2 = \frac{a^3 - a^2x}{x} = \frac{a^3}{x} - a^2$$

i.e. $y^2 = a^3/x - a^2 \longrightarrow (1)$ 1 M

Differentiating eqn (1) wrt 'x',

$$2yy_1 = -\frac{a^3}{x^2}$$ 1 M

i.e. $$y_1 = -\frac{a^3}{2yx^2}$$ 1 M

At $(a,0) \Rightarrow$ y_1 becomes infinity and hence we have to consider dx/dy.

Let, $$x_1 = \frac{dx}{dy} = -\frac{2yx^2}{a^3}$$ 1 M

and $x_1 = 0$ at $(a,0)$.

Now, $$x_2 = -\frac{2}{a^3}\{y \times 2x x_1 + x^2\}$$

$$x_2 = -\frac{2}{a^3}\{0 + a^2\}$$ 1 M

At $(a,0)$ $x_2 = -\frac{2}{a}$

We have, $$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$ 1 M

$$= \left|\frac{(1+0)^{3/2}}{-2/a}\right| = \left|-\frac{a}{2}\right|$$

Thus, $\rho = a/2$ 1 M

Total $\longrightarrow$ 7 M

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Q. No. 3 a. Expand $\sqrt{1+\sin 2x}$ by Maclaurin's series upto the term containing x^5.

Sol^n: Maclaurin's series formula is,

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + y_4(0)\frac{x^4}{4!} + \frac{x^5}{5!}y_5(0) + \cdots \quad \text{1M}$$

Let, $y = \sqrt{1+\sin 2x}$

$$= \sqrt{\cos^2 x + \sin^2 x + 2\sin x\cos x}$$

$$= \sqrt{(\cos x + \sin x)^2}$$

$$y = \cos x + \sin x \qquad \therefore y(0) = 1$$

Differentiating wrt 'x',

$$y_1 = -\sin x + \cos x \qquad \therefore y_1(0) = 1 \quad \text{2M}$$

$$y_2 = -\cos x - \sin x$$

$$= -(\cos x + \sin x)$$

$$y_2 = -y \qquad \therefore y_2(0) = -1$$

Differentiating wrt 'x',

$$y_3 = -y_1 \qquad y_3(0) = -1 \quad \text{1M}$$

$$y_4 = -y_2 \qquad y_4(0) = -(-1) = 1$$

$$y_5 = -y_3 \qquad y_5(0) = -(-1) = 1 \quad \text{1M}$$

Thus, by substituting these values in the expansion of $y(x)$ we get

$$\sqrt{1+\sin 2x} = 1 + x + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}\times 1 + \frac{x^5}{5!}\times(1) + \cdots \quad \text{1M}$$

$$= 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

1M

Total → 7M

3b. If $u = \tan^{-1}(y/x)$, where $x = e^t - e^{-t}$ and $y = e^t + e^{-t}$, find the total derivative du/dt using partial differentiation.

Sol^n The total derivative rule is,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$ 1M

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{-y}{x^2} = \frac{x^2}{x^2+y^2} \times \frac{-y}{x^2}$$

$$\frac{\partial u}{\partial x} = -\frac{y}{x^2+y^2}$$ 1M

Similarly, $\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x}$

$$= \frac{x^2}{x^2+y^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$ 1M

$$\therefore \frac{du}{dt} = -\frac{y}{x^2+y^2} \times (e^t + e^{-t}) + \frac{x}{x^2+y^2} \times (e^t - e^{-t})$$ 1M

$$= -\frac{(e^t+e^{-t})(e^t+e^{-t})}{(e^t-e^{-t})^2+(e^t+e^{-t})^2} + \frac{(e^t-e^{-t})(e^t-e^{-t})}{(e^t-e^{-t})^2+(e^t+e^{-t})^2}$$

$$= \frac{-(e^t+e^{-t})^2 + (e^t-e^{-t})^2}{(e^t+e^{-t})^2 + (e^t-e^{-t})^2}$$ 2M

$$\therefore \frac{du}{dt} = \frac{-4}{2e^{2t}+2e^{-2t}} = \frac{-2}{e^{2t}+e^{-2t}}$$ 1M

Total → 7M

3c. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $\omega = \frac{xy}{z}$, show that

$\frac{\partial(u,v,\omega)}{\partial(x,y,z)} = 4.$

Sol^n: Given, $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $\omega = \frac{xy}{z}$

$$\frac{\partial(u,v,\omega)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$ 2M

$$= -\frac{yz}{x^2}\left\{\left(\frac{-zx}{y^2} \times \frac{-xy}{z^2}\right) - \left(\frac{x}{z}\right)\left(\frac{x}{y}\right)\right\}$$

$$-\frac{z}{x}\left\{\frac{z}{y}\left(\frac{-xy}{z^2}\right) - \frac{y}{z}\cdot\frac{x}{y}\right\} + \frac{y}{x}\left\{\frac{z}{y}\times\frac{x}{z} - \frac{y}{z}\left(\frac{-zx}{y^2}\right)\right\}$$

2M

$$= -\frac{yz}{x^2}\left\{\frac{x^2}{yz} - \frac{x^2}{yz}\right\} - \frac{z}{x}\left\{\frac{-x}{z} - \frac{x}{z}\right\} + \frac{y}{x}\left\{\frac{x}{y} + \frac{x}{y}\right\}$$

1M

$$= 0 - \frac{z}{x}\left\{-\frac{2x}{z}\right\} + \frac{y}{x}\left\{\frac{2x}{y}\right\}$$ 1M

$$= 0 + 2 + 2 = 4$$

$\therefore \frac{\partial(u,v,\omega)}{\partial(x,y,z)} = 4$ 1M

Total → 7M

4a. Evaluate (i) $\lim_{x\to 0} (a^x + x)^{1/x}$

(ii) $\lim_{x\to \pi/2} (\tan x)^{\tan 2x}$

Sol^n (i) Let, $k = \lim_{x\to 0} (a^x + x)^{1/x} \dashrightarrow 1^\infty$

$\log_e k = \lim_{x\to 0} \frac{\log(a^x + x)}{x} \dashrightarrow \left(\frac{0}{0}\right)$ 1M

Applying L'Hospital's Rule,

$= \lim_{x\to 0} \frac{\frac{1}{a^x + x} \times (a^x \log a + 1)}{1}$ 1M

$= \lim_{x\to 0} \frac{a^x \log a + 1}{a^x + x} \dashrightarrow$ 1M

$\log_e k = \log a + 1$ 1M

$\Rightarrow k = e^{(\log a + 1)} = e^{\log a} . e$ 1M

$\underline{\underline{k = ae}}$ 1M

Total $\rightarrow$ 3M

(ii) $k = \lim_{x\to \pi/4} (\tan x)^{\tan 2x} \dashrightarrow 1^0$

$\hookrightarrow$ 1M

$\log_e k = \lim_{x\to \pi/4} \tan 2x \log \tan x \dashrightarrow \infty \times 0$

$= \lim_{x\to \pi/4} \frac{\log \tan x}{\cot 2x} \dashrightarrow 0/0$

1M

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$$= \lim_{x \to \pi/4} \frac{\sec^2 x / \tan x}{-2 \operatorname{cosec}^2 x}$$

by Applying L' Hospital's Rule

$$= -\frac{1}{2} \lim_{x \to \pi/4} \frac{\sec^2 x}{\tan x \operatorname{cosec}^2 x} \qquad 1M$$

$$= -\frac{1}{2} \times \frac{2}{1} = -1$$

$$\log_e k = -1$$

$$\Rightarrow k = e^{-1}$$

$$\underline{\underline{k = 1/e}} \qquad 1M \ (3M)$$

Total → 6M

4b. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that

$$x^2 \partial u/\partial x + y^2 \partial u/\partial y + z^2 \partial u/\partial z = 0.$$

Sol^n: Let, $u = f(s,t)$, where

$$s = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$$

and $t = \dfrac{z-x}{xz} = \dfrac{1}{x} - \dfrac{1}{z}$

$$\left.\begin{aligned} \frac{\partial s}{\partial x} &= -\frac{1}{x^2}, \quad \frac{\partial s}{\partial y} = \frac{1}{y^2}, \quad \frac{\partial s}{\partial z} = 0 \\ \frac{\partial t}{\partial x} &= -\frac{1}{x^2}, \quad \frac{\partial t}{\partial y} = 0, \quad \frac{\partial t}{\partial z} = \frac{1}{z^2} \end{aligned}\right\} 2M$$

Now by the chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} = -\frac{1}{x^2}\left(\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\right)$$

1M

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} = \frac{1}{y^2}\cdot\frac{\partial u}{\partial s} \quad 1M$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z} = \frac{1}{z^2}\cdot\frac{\partial u}{\partial t} \quad 1M$$

These give,

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\left(\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right) + \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \quad 1M$$

$$= 0 = RHS \quad 1M$$

Total → 7M

4c. Find the extreme values of x^3+y^3-3axy, $a \geq 0$.

Sol^n Let, $f(x,y) = x^3+y^3-3axy$

$f_x = 3x^2-3ay$, $f_y = 3y^2-3ax$

Let, $A = f_{xx}$ $B = f_{xy}$ & $C = f_{yy}$ 1M

Considering necessary & sufficient conditions

$f_x = 0$ and $f_y = 0$

$3x^2-3ay = 0 \Rightarrow x^2 = ay \rightarrow$(i)

and $3y^2-3ax = 0 \Rightarrow y^2 = ax \rightarrow$(ii) 1M

on squaring eqn (i),

$x^4 = a^2y^2$

$\therefore x^4 = a^2(ax)$ using eqn (ii)

$x^4 = a^3x$

$x^3 = a^3 \Rightarrow \boxed{x=a}$

Substituting in eqn (i),

$\therefore a^2 = ay \quad \therefore \boxed{y=a}$

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$A = \frac{\partial^2 u}{\partial x^2} = 6x = 6a > 0$

$B = f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(3y^2 - 3ax)$

$= -3a < 0$ 1M

$C = f_{yy} = 6y = 6a > 0$

(a, a) is one of the stationary points

Again solving equations (i) & (ii),

$x^2 - ay = 0$ & $y^2 - ax = 0$

$y^2 - ax = 0 \Rightarrow x = \cancel{\frac{a^2}{y}}\ \frac{y^2}{a}$

Using $x = \frac{y^2}{a}$ in $x^2 - ay = 0$

$$\left(\frac{y^2}{a}\right)^2 - ay = 0$$

$\Rightarrow y^4 - a^3 y = 0$ or $y(y^3 - a^3) = 0$

$\Rightarrow y = 0$ & $y^3 - a^3 = 0 \Rightarrow y = a$

From eqn $x^2 - ay = 0$, we get $x = 0$ when $y = 0$ and when $y = a \Rightarrow x = \pm a$. 1M

But, $x = -a, y = a$ do not satisfy eqn $y^2 - ax = 0$. hence they are not solution.

At $x = 0, y = 0$ $A = 0, B = -3a, C = 0$

$AC - B^2 = 0 - (-3a)^2 = -9a^2 < 0$ 1M

& hence there is neither maximum nor minimum at $x = 0$ & $y = 0$. 1M

At $x = a$ & $y = a$, $A = 6a$, $B = -3a$, $C = 6a$

$\therefore AC - B^2 = 36a^2 - (-3a)^2 = 36a^2 - 9a^2 > 0$

Also, $A = 6a > 0$, if $a > 0$ & $A \not> 0$, if $a < 0$.

Hence, there is maximum or minimum according to $a < 0$ or $a > 0$. 1M

Total $\rightarrow$ 7M

TOPIC Calculus & Differential Equations DATE 13/03/2022

Q 5 a. Solve : $\frac{dy}{dx} + y \tan x = y^3 \sec x$

Sol^n Dividing by y^3, the given equation becomes

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\tan x = \sec x \longrightarrow (i) \qquad 1M$$

This is a Bernoulli equation. Let $\frac{1}{y^2} = v$, so

that, $-\frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$. 1M

Equation (i) becomes

$$\frac{dv}{dx} - 2(\tan x)v = -2\sec x \longrightarrow (ii) \qquad 1M$$

This is a linear equation in v, with $P = -2\tan x$ and $Q = -2\sec x$.

Integrating factor (I.F.) $= e^{\int P\,dx}$

$$\int P\,dx = -\int 2\tan x\,dx = 2\log\cos x$$
$$= \log\cos^2 x$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\log\cos^2 x} = \cos^2 x \qquad 1M$$

Hence, the general solution of equation (ii) is

$$v.\cos^2 x = \int(-2\sec x)\cos^2 x\,dx + c$$

$$= -2\int\cos x\,dx + c = 2\sin x + c \qquad 1M$$

$$\frac{1}{y^2}.\cos^2 x = 2\sin x + c$$

$$\underline{\underline{\cos^2 x = y^2(2\sin x + c)}} \qquad 1M$$

This is the general solution of the given equation.

Total → 6M

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5b. Water at temperature 10°c takes 5 minutes to warm up ato 20°C at a room temperature of 40°C. Find the temperature of the water after 20 minutes.

Solⁿ The Newton's Law of cooling is

$$T = t_2 + (t_1 - t_2)e^{-kt}$$ 1M

$t_2 = 40°C$ (Room temperature)

$t_1 = 10°C$, $t = 5$ minutes

Newton's Law of cooling is,

$$T = t_2 + (t_1 - t_2)\, e^{-kt}$$

$$20 = 40 + (-30)e^{-5k}$$ 1M

$$e^{5k} = \frac{30}{20} = 1.5$$ 1M

$$5k = \log(1.5)$$

$$\Rightarrow k = \frac{\log(1.5)}{5} = 0.08109$$ 1M

Temperature after 20 minutes, T = ? at $t = 20$ minutes

$$T = t_2 + (t_1 - t_2)e^{-kt}$$

$$= 40 + (10-40)\, e^{-20(0.08109)}$$ 1M

$$T = (34.07)°C$$ → 2M

Total → 7M

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5c. Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

Sol^n The given equation can be rewritten as

$p(p^2 + 2xp - y^2p - 2xy^2) = 0$

or $p\{p(p+2x) - y^2(p+2x)\} = 0$

or $p(p-y^2)(p+2x) = 0$ 1M

or $p = 0,\ p - y^2 = 0,\ p + 2x = 0$ 1M

Integrating,

$\frac{dy}{dx} = 0, \Rightarrow y = c \longrightarrow (i)$ 1M

$p - y^2 = 0 \Rightarrow \frac{dy}{dx} - y^2 = 0$

$\Rightarrow \frac{dy}{y^2} - dx = 0$

$\int \frac{dy}{y^2} - \int 1 . dx = c$

$-y^{-1} - x = c \Rightarrow x + \frac{1}{y} = c \longrightarrow (ii)$ 1M

Next, consider $p + 2x = 0$

$\frac{dy}{dx} + 2x = 0$

$dy + 2x\,dx = 0$ (separating the variables)

$\int dy + 2\int x\,dx = c$

$\Rightarrow y + x^2 = c \longrightarrow (iii)$ 2M

Hence, the general solution is

$(y-c)(y+x^2-c)\left(x+\frac{1}{y}-c\right) = 0$ 1M

Total $\rightarrow$ 7M

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Q. 06 a. Solve $(x^2+y^3+6x)\,dx + y^2x\,dy = 0$

Sol^n The equation is of the form $M\,dx + N\,dy = 0$, with $M = x^2+y^3+6x$ & $N = y^2x$

$\frac{\partial M}{\partial y} = 3y^2 \qquad \frac{\partial N}{\partial x} = y^2$ 1M

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so that the equation is not exact.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3y^2 - y^2 = 2y^2$, which is close to N term. 1M

$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y^2x}(2y^2) = \frac{2}{x} = f(x)$

I.F. $= e^{\int f(x)\,dx} = e^{\int 2/x\,dx} = e^{2\log x}$

$= x^2$ 1M

Multiplying the equation by this factor, we get the exact equation

$(x^4 + x^2y^3 + 6x^3)\,dx + y^2x^3\,dy = 0$ 1M

The general solution of this exact equation is

$\int x^4\,dx + y^3\int x^2\,dx + 6\int x^3\,dx = c$ 1M

$x^5/5 + y^3\frac{x^3}{3} + 6\frac{x^4}{4} = c$

$6x^5 + 10y^3x^3 + 45x^4 = 30c$ 1M

Total → 6M

6b Prove that the system of parabolas $y^2 = 4a(x+a)$ are self orthogonal.

Sol^n The given family is,

$$y^2 = 4a(x+a) \longrightarrow (i)$$

Differentiating eqn (i) w.r.t. 'x', we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow a = \frac{1}{2}y\frac{dy}{dx}$$ 1M

Substituting this value of 'a' in eqn (i)

$$y^2 = 4 \times \frac{1}{2} y \times \frac{dy}{dx}\left(x + \frac{y}{2} \times \frac{dy}{dx}\right)$$

$$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$ 1M

$$y = 2xy\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 \longrightarrow (ii)$$ 1M

This is the differential equation of the given family.

Changing dy/dx to $-(dx/dy)$ in (ii),

$$y = -2x\frac{dx}{dy} + y\left(\frac{dx}{dy}\right)^2$$ 1M

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y \longrightarrow (iii)$$ 1M

This is the differential equation of the orthogonal trajectories. This equation is identical to eqn (ii). Hence, the given family and the family of its orthogonal trajectories are not different, that is the given family is self-orthogonal. 2M

Total $\rightarrow$ 7M

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6c. Find the general solution and singular solutions of $xp^2+xp-yp+1-y=0$.

Sol^n: Given equation is,

$$xp^2+xp-yp+1-y=0$$

$$\Rightarrow \quad xp^2+xp+1 = yp+y = y(p+1)$$

$$y = \frac{xp^2+xp+1}{p+1}$$ 1M

$$= \frac{px(p+1)}{p+1} + \frac{1}{p+1}$$

$$y = px + \frac{1}{p+1},$$ which is in the Clairaut's form 1M

$$y = px + f(p)$$

Hence, the solution is $y = cx + \frac{1}{c+1} \longrightarrow ①$ 1M

Differentiating eqn ① partially wrt 'c',

$$0 = x - \frac{1}{(c+1)^2} \Rightarrow x = \frac{1}{(c+1)^2}$$

$$\Rightarrow (c+1)^2 = 1/x$$

$$c+1 = \frac{1}{\sqrt{x}} \Rightarrow c = \frac{1}{\sqrt{x}} - 1$$ 2M

Substituting c value in eqn ①,

$$y = \left(\frac{1}{\sqrt{x}} - 1\right)x + \frac{1}{\frac{1}{\sqrt{x}} - 1 + 1}$$ 1M

$$= \sqrt{x} - x + \frac{1}{1/\sqrt{x}}$$

$$y = \sqrt{x} - x + \sqrt{x}$$

$$\Rightarrow x+y = 2\sqrt{x}$$

$$(x+y)^2 = 4x \quad \text{(squaring)}$$ 1M

is the required singular solution. Total → 7M

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Q 7 a Solve $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$

Sol^n By inspection $D = -1$ is one of the root of the given equation. 1M

By synthetic division,

-1	4	-8	-7	11	6
		-4	12	-5	-6
	4	-12	5	6	0

1M

$\therefore\ 4D^3 - 12D^2 + 5D + 6 = 0 \longrightarrow (i)$

$D = 2$ is a root of eqn (i) by inspection.

Using synthetic division,

2	4	-12	5	6
		8	-8	-6
	4	-4	-3	0

$\therefore\ 4D^2 - 4D - 3 = 0 \longrightarrow (ii)$ 1M

Factorizing eqn (ii),

$4D^2 - 6D + 2D - 3 = 0$

$2D(2D-3) + 1(2D-3) = 0$

$(2D+1)(2D-3) = 0$

$\Rightarrow\ 2D + 1 = 0$ & $2D - 3 = 0$

$D = -\frac{1}{2}$ $\Rightarrow D = \frac{3}{2}$

$\therefore$ The roots are,

$D = -1, -\frac{1}{2}, 2, \frac{3}{2}$ 2M

$\therefore$ The solution is,

$y = c_1 e^{-x} + c_2 e^{-x/2} + c_3 e^{2x} + c_4 e^{3x/2}$

1M

Total $\rightarrow$ 6M

7b Solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2+2x+4$

Sol^n Given equation can be rewritten in the form

$$D^2y + Dy = x^2+2x+4$$

$$(D^2+D)y = x^2+2x+4$$

The auxilliary equations is,

$$D^2+D=0 \Rightarrow D(D+1)=0 \Rightarrow D=0 \ \&$$

$$D+1=0 \Rightarrow D=-1$$

$$\therefore D = 0, -1 \qquad 1M$$

$$\therefore C.F. = c_1 + c_2 e^{-x} \qquad 1M$$

To find the particular integral

$$P.I. = \frac{1}{D^2+D}(x^2+2x+4) \qquad 1M$$

$$= \frac{1}{D(D+1)}(x^2+2x+4)$$

$$= \frac{1}{D}(D+1)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1-D+D^2-----)(x^2+2x+4) \quad 1M$$

$$= \frac{1}{D}\left[(x^2+2x+4) - D(x^2+2x+4) + D^2(x^2+2x+4) + 0\right]$$

$$= \frac{1}{D}\left[x^2+2x+4-(2x+2)+2\right] \qquad 1M$$

$$= \frac{1}{D}\left[x^2+4\right] = \int x^2\,dx + \int 4\,dx$$

$$= \frac{x^3}{3} + 4x \qquad 1M$$

$\therefore$ The complete solution is,

$$y = c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x \qquad 1M$$

Total $\rightarrow$ 7M

7c. Using the method of variation of parameters, solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

Sol^n Given equation is,

$$(D^2-6D+9)\,y = e^{3x}/x^2$$

Auxilliary equation is,

$$D^2-6D+9=0$$

$$(D-3)^2 = 0 \Rightarrow D = 3, 3$$

$$C.F. = e^{3x}(c_1+c_2x) \qquad 1M$$

Particular integral,

$$P.I. = (A+Bx)e^{3x}$$

where $y_1 = e^{3x}$ and $y_2 = xe^{3x}$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \qquad 1M$$

$$= e^{3x}(3xe^{3x}+e^{3x}) - xe^{3x}(3e^{3x})$$

$$= e^{6x}(3x+1) - 3xe^{6x} = 3xe^{6x}+e^{6x}-3xe^{6x}$$

$$= e^{6x} \qquad 1M$$

Now, $A = -\int \frac{y_2X}{W}\,dx$

$$= -\int \frac{xe^{3x}}{e^{6x}}\left(\frac{e^{3x}}{x^2}\right)dx = -\int \frac{1}{x}\,dx$$

$$= -\log x \qquad 1M$$

$$B = \int \frac{y_1X}{W}\,dx = \int \frac{e^{3x}}{e^{6x}} \times \frac{e^{3x}}{x^2}\,dx$$

$$= \int \frac{1}{x^2}\, dx = -\frac{1}{x}$$ 1M

$\therefore$ P.I. $= (A + Bx)e^{3x}$

$$= \left(-\log x - \frac{1}{x} \times x\right) e^{3x}$$

$$= -(1 + \log x)e^{3x}$$ 1M

Hence, the complete solution is,

$y =$ C.F. $+$ P.I.

$y = (c_1 + c_2 x)e^{3x} - (1 + \log x)e^{3x}$ 1M

Total → 7M

8a Solve $\left(\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y\right) = e^{2x}$

Solⁿ. $(D^3 - 5D^2 + 7D - 3)y = e^{2x}$

Auxilliary equation is,

$$D^3 - 5D^2 + 7D - 3 = 0$$

$D = 1$ is a root by inspection method.

1	1	-5	7	-3
		1	-4	3
	1	-4	3	0

$\therefore D^2 - 4D + 3 = 0$ 1M

$D^2 - 3D - D + 3 = 0$

$D(D-3) - 1(D-3) = 0$

$\Rightarrow (D-1)(D-3) = 0 \Rightarrow D = 1, 3$ 1M

$\therefore D = 1, 1, 3$

$\therefore$ C.F. $= (c_1 + c_2 x)e^{x} + c_3 e^{3x}$ 1M

$$P.I. = \frac{\phi(x)}{f(D)} = \frac{e^{2x}}{D^3 - 5D^2 + 7D - 3}$$ 1M

put $D = 2$

$$= \frac{e^{2x}}{(2)^3 - 5(2)^2 + 7(2) - 3}$$

$$= \frac{e^{2x}}{-1} = -e^{2x}$$ 1M

$\therefore P.I. = -e^{+2x}$

The complete solution is,

$y = C.F. + \text{Particular Integral}$

$$y = (c_1 + c_2 x)e^x + c_3 e^{3x} - e^{2x}$$ 1M

Total - 4M

8b. Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = \sin x$

Sol^n $(D^2 + 4D + 3)y = \sin x$

A.E. is $D^2 + 4D + 3 = 0$

$D^2 + 3D + D + 3 = 0$

$D(D+3) + 1(D+3) = 0$

$(D+1)(D+3) = 0 \Rightarrow D = -1, -3$

$\therefore C.F. = c_1 e^{-x} + c_2 e^{-3x}$ 1M

$$P.I. = \frac{\phi(X)}{f(D)} = \frac{\sin x}{D^2 + 4D + 3}$$ 1M

put, $D^2 = -a^2 = -1$

$$= \frac{\sin x}{-1 + 4D + 3} = \frac{\sin x}{4D + 2}$$ 1M

$$= \frac{1}{2} \times \frac{\sin x}{(2D+1)}$$

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$$= \frac{1}{2} \times \frac{(2D-1)\sin x}{(2D-1)(2D+1)}$$ 1M

$$= \frac{1}{2} \times \frac{\{2D(\sin x) - \sin x\}}{4D^2 - 1}$$

$$= \frac{1}{2} \times \frac{(2\cos x - \sin x)}{4(-1) - 1}$$ 1M

$$= \frac{2\cos x - \sin x}{-10}$$ 1M

$\therefore$ The complete solution is,

$y = C.F. + P.I.$

$$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{10}(2\cos x - \sin x)$$ 1M

Total → 7M

8c. Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin[2\log(1+x)]$

Sol^n: The given equation is a Legendre equation in which $a=1$ and $b=1$.

put $ax+b = (1+x) = e^z$ that is

$\log(1+x) = z$, so that

$$\left.\begin{aligned} (1+x)\frac{dy}{dx} &= aDy = Dy \quad [\because a=1] \\ (1+x)^2\frac{d^2y}{dx^2} &= a^2D(D-1)y = D(D-1)y \end{aligned}\right\} 2M$$

Substituting these in the given equation,

$D(D-1)y + Dy + y = \sin 2z$

$$(D^2+1)y = \sin 2z$$

C.F. $\Rightarrow$ $D^2+1=0 \Rightarrow D^2=-1 \Rightarrow D=\pm i$

C.F. $= (c_1 \cos z + c_2 \sin z)$ 1M

$= [c_1 \cos \log(1+x) + c_2 \sin \log(1+x)]$ 1M

To find Particular integral,

$$P.I. = \frac{\sin 2z}{D^2+1}$$ 1M

put $D^2=-4 \Rightarrow P.I. = \frac{\sin 2z}{-4+1}$ 1M

$$= -\frac{1}{3} \sin 2z$$

$$= -\frac{1}{3} \sin[2\log(1+x)]$$

The complete solution is,

$$y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \frac{1}{3}\sin[2\log(1+x)]$$ 1M

Total $\rightarrow$ 7M

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Q. 09 a) Find the rank of the matrix

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Solⁿ Interchanging rows R_1 & R_4

$R_1 \Leftrightarrow R_4$ $A \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$ 1M

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$A \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 2 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$ 2M

$R_3 \to R_3 - 2R_2$, $R_4 \to R_4 + R_1$

$A \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 2M

$\therefore \rho(A) = 3$ 1M

Total → 6M

9b. Solve the system of equations by using the Gauss elimination method.

$3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$.

Solⁿ The augmented matrix is,

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{array}\right] = [A : B]$$

$R_1 \longleftrightarrow R_3$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$[A : B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{array}\right]$$ 2M

$R_3 \to 7R_3 - 5R_2$

$$[A : B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{array}\right]$$ 2M

By back substitution,

$8z = -8 \Rightarrow z = -1$ 1M

$-7y - 3z = -11 \Rightarrow 7y + 3z = 11$

$7y - 3 = 11 \Rightarrow 7y = 14$

$\Rightarrow y = 2$ 1M

$x + 2y + z = 4$

$x + 4 - 1 = 4 \Rightarrow x + 3 = 4 \Rightarrow x = 1$ 1M

$\therefore x = 1, y = 2, z = -1$

Total → 7M

9c. Using the Gauss-Seidel iteration method, solve the equations

$$83x + 11y - 4z = 95$$
$$3x + 8y + 29z = 71$$
$$7x + 52y + 13z = 104$$

Carry out four iterations, starting with the initial approximations (0, 0, 0).

Sol^n Rewriting the equations, so that they become diagonally dominant

$$\left.\begin{aligned} 83x + 11y - 4z &= 95 \\ 7x + 52y + 13z &= 104 \\ 3x + 8y + 29z &= 71 \end{aligned}\right\} \text{1M}$$

Hence,

$$\left.\begin{aligned} x &= \frac{1}{83}[95 - 11y + 4z] \\ y &= \frac{1}{52}[104 - 7x - 13z] \\ z &= \frac{1}{29}[71 - 3x - 8y] \end{aligned}\right\} \text{1M}$$

1st iteration : $x = 0,\ y = 0,\ z = 0$

$$x^{(1)} = \frac{1}{83}[95 - 11 \times 0 + 4 \times 0] = 1.14457$$

$$y^{(1)} = \frac{1}{52}[104 - 7(1.14457) - 13(0)]$$
$$= 1.84592$$

$$z^{(1)} = \frac{1}{29}[71 - 3(1.14457) - 8(1.84592)]$$
$$= 1.82065$$

1M

2nd iteration : $x = 1.14457,\ y = 1.84592,\ z = 1.82065$

$$x^{(2)} = \frac{1}{83}[95 - 11(1.84592) + 4(1.82065)]$$

$$= 0.98768$$

$$y^{(2)} = \frac{1}{52}[104 - 7(0.98768) - 13(1.82065)]$$

$$= 1.41188$$

$$z^{(2)} = \frac{1}{29}[71 - 3(0.98768) - 8(1.41188)]$$

$$= 1.95662$$

1M

3rd iteration : $x = 0.98768,\ y = 1.41188,\ z = 1.95662$

$$x^{(3)} = \frac{1}{83}[95 - 11(1.41188) + 4(1.95662)]$$

$$= 1.05176$$

$$y^{(3)} = \frac{1}{52}[104 - 7(1.05176) - 13(1.95662)]$$

$$= 1.36926$$

$$z^{(3)} = \frac{1}{29}[71 - 3(1.05176) - 8(1.36926)]$$

$$= 1.96175$$

1M

4th iteration : $x = 1.05176,\ y = 1.36926,\ z = 1.96175$

$$x^{(4)} = \frac{1}{83}[95 - 11(1.36926) + 4(1.96175)]$$

$$= 1.05765$$

$$y^{(4)} = \frac{1}{52}[104 - 7(1.05765) - 13(1.96175)]$$

$$= 1.36719$$

$$z^{(4)} = \frac{1}{29}\left[71 - 3(1.05765) - 8(1.36719)\right]$$

$$z^{(4)} = 1.96171 \qquad 1M$$

$\therefore x = 1.05765,\ y = 1.36719,\ z = 1.96171$ 1M

Total → 7M

Q 10a Test for consistency and solve

$5x+3y+7z=4;\ 3x+26y+2z=9;\ 7x+2y+10z=5$

Sol^n The augmented matrix is,

$$[A:B] = \left[\begin{array}{ccc:c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$R_1 \to R_1/5$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & \frac{3}{5} & \frac{7}{5} & \frac{4}{5} \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right] \qquad 1M$$

$R_2 \to R_2 - 3R_1,\quad R_3 \to R_3 - 7R_1$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 3/5 & 7/5 & 4/5 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & -11/5 & 1/5 & -3/5 \end{array}\right] \qquad 1M$$

$R_3 \to 11R_3 + R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 3/5 & 7/5 & 4/5 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad 1M$$

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The matrix is in Echelon form. It has two non-zero rows.

$\therefore \rho(A) = \rho[A \vdots B] = 2 < r$, number of unknowns 1M

Hence, the given equations are consistent and have infinitely many solutions.

$$x + \frac{3}{5}y + \frac{7}{5}z = 4/5 \longrightarrow (1)$$

$$\frac{121}{5}y + \frac{11}{5}z = \frac{33}{5}$$

$$\Rightarrow \frac{11}{5}y = \frac{1}{5}z + \frac{3}{5}$$

$$11y = z + 3 \Rightarrow y = \frac{z+3}{11} \rightarrow (2) \quad 1M$$

Using eqn (2) in eqn (1),

$$x = \frac{4}{5} - \frac{3}{5}y - \frac{7}{5}z$$

$$= \frac{4}{5} - \frac{3}{5}\left(\frac{z+3}{11}\right) - \frac{7}{5}z$$

$$x = \frac{4}{5} - \frac{3z}{55} - \frac{9}{55} - \frac{7}{5}z$$

$$x = -\frac{16z}{11} + \frac{7}{11} \quad 1M$$

Let us take $z = k$, $k \in R$.

$$y = \frac{k+3}{11}, \quad x = -\frac{16}{11}k + \frac{7}{11}$$

By giving different value for k, we get different solutions. Thus, the solutions of given system of equations are given by

$$x = \frac{1}{11}(7-16k), \quad y = \frac{1}{11}(3+k), \quad z = k$$

Total → 6M

10 b. Using the Gauss Jordan method, solve
$x+y+z=11$; $3x-y+2z=12$; $2x+y-z=3$.

Sol^n: The augmented matrix is,

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 3 & -1 & 2 & 12 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & -4 & -1 & -21 \\ 0 & -1 & -3 & -19 \end{array}\right]$$ 2M

$R_1 \to 4R_1 + R_2$, $R_3 \to 4R_3 - R_2$

$$[A:B] \sim \left[\begin{array}{ccc|c} 4 & 0 & 3 & 23 \\ 0 & -4 & -1 & -21 \\ 0 & 0 & -11 & -55 \end{array}\right]$$ 1M

$R_3 \to R_3 \times \left(-\frac{1}{11}\right)$

$$[A:B] \sim \left[\begin{array}{ccc|c} 4 & 0 & 3 & 23 \\ 0 & -4 & -1 & -21 \\ 0 & 0 & 1 & 5 \end{array}\right]$$ 1M

$R_1 \to R_1 - 3R_3$, $R_2 \to R_2 + R_3$

$$[A:B] \sim \left[\begin{array}{ccc|c} 4 & 0 & 0 & 8 \\ 0 & -4 & 0 & -16 \\ 0 & 0 & 1 & 5 \end{array}\right]$$ 1M

$$\left.\begin{array}{l} \text{By back substitution, } 4x = 8 \Rightarrow x = 2 \\ -4y = -16 \Rightarrow y = 4 \\ z = 5 \end{array}\right\} 2M$$

$\therefore x = 2,\ y = 4,\ z = 5$

Total → 7M

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10 c. Find the largest eigenvalue and the corresponding eigenvector of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ with the initial approximate eigenvector $[1\ 0\ 0]^T$.

Sol^n The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

on expanding we have

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$ 1M

$\lambda = 2$ is a root by inspection. 1M

By synthetic division

2	1	-12	36	-32
	0	2	-20	32
	1	-10	16	0

$\therefore \lambda^2 - 10\lambda + 16 = 0$ 1M

$(\lambda-2)(\lambda-8) = 0 \Rightarrow \lambda = 2, 8$

$\therefore \lambda = 2, 2, 8$ are the eigenvalues

$$(6-\lambda)x - 2y + 2z = 0$$
$$-2x + (3-\lambda)y - 1z = 0$$
$$2x - 1y + (3-\lambda)z = 0$$

Case (i): Let $\lambda = 2$.

$$4x - 2y + 2z = 0$$
$$-2x + y - z = 0$$ 1M
$$2x - y + z = 0$$

The above set of equations are all same as we have only one independent equation $2x - y + z = 0$ & hence we can choose two variables arbitrarily.

Let $z = k_1$ and $y = k_2$
$\therefore \; x = (k_2 - k_1)/2$ } 1M

$\therefore X_1 = \left[\dfrac{k_2 - k_1}{2}, k_2, k_1\right]$ is the eigenvector corresponding to $\lambda = 2$ where k_1, k_2 are not simultaneously equal to zero.

Case (ii): Let $\lambda = 8$ & we have

$$-2x - 2y + 2z = 0$$
$$-2x - 5y - 1z = 0 \qquad 1M$$
$$2x - 1y - 5z = 0$$

$$\frac{x}{\begin{vmatrix} -5 & -1 \\ -1 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -1 \\ 2 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -5 \\ 2 & -1 \end{vmatrix}}$$

$$\frac{x}{25-1} = \frac{-y}{10+2} = \frac{z}{2+10}$$

$$\frac{x}{24} = \frac{-y}{12} = \frac{z}{12}$$

$$\Rightarrow \frac{x}{12} = \frac{-y}{6} = \frac{z}{6} \text{ or } \frac{x}{2} = \frac{y}{-1} = \frac{z}{1} \qquad 1M$$

$\Rightarrow X_2 = (2, -1, 1)'$ is the eigenvector corresponding to $\lambda = 8$.

Total → 7M

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