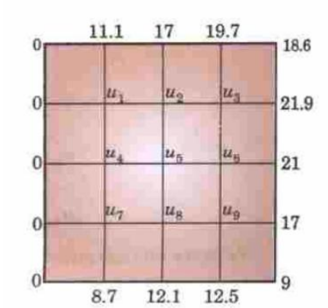
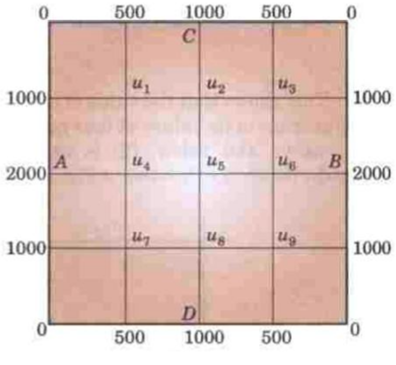
Karnatak Law Society's
Vishwanathrao Deshpande Institute of Technology, Haliyal - 581 329

Model Question Paper-I with effect from 2021 (CBCS Scheme)
Third Semester B.E Degree Examination
Transform Calculus, Fourier Series and Numerical Techniques (21MAT31)

TIME: 03 Hours **Max. Marks: 100**
Faculty Name : Prof. Vijaya T. Chitrali

Note: Answer any FIVE full questions, choosing at least ONE question from each MODULE.

<table>
<tr><th colspan="3">Module - 1</th><th>Marks</th></tr>
<tr><td>1</td><td>a</td><td>Evaluate (i) $L\left\{\frac{\cos2t-\cos3t}{t}\right\}$ (ii) $L(t^2e^{-3t}\sin2t)$</td><td>06</td></tr>
<tr><td></td><td>b</td><td>If $f(t)=\begin{cases} t, & 0\le t\le a\\ 2a-t, & a\le t\le 2a\end{cases}$, $f(t+2a)=f(t)$ then show that $L[f(t)]=\frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$</td><td>07</td></tr>
<tr><td></td><td>c</td><td>Solve by using Laplace Transforms, $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=e^{-t}, y(0)=0, y'(0)=0.$</td><td>07</td></tr>
<tr><td colspan="4">OR</td></tr>
<tr><td>2</td><td>a</td><td>Evaluate $L^{-1}\left[\frac{4s+5}{(s+1)^2(s+2)}\right]$</td><td>06</td></tr>
<tr><td></td><td>b</td><td>Evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ by using convolution theorem.</td><td>07</td></tr>
<tr><td></td><td>c</td><td>Express $f(t)=\begin{cases}\sin t & 0\le t<\pi\\ \sin2t & \pi\le t<2\pi\\ \sin3t & t\ge2\pi\end{cases}$ in terms of unit step function and find its Laplace transform.</td><td>07</td></tr>
<tr><td colspan="4">Module - 2</td></tr>
<tr><td>3</td><td>a</td><td>Obtain the Fourier Series for the function $f(x)=|x|$ in $(-\pi,\pi)$.</td><td>06</td></tr>
<tr><td></td><td>b</td><td>Expand $f(x)=\frac{(\pi-x)^2}{4}$ as a Fourier Series in the interval $(0,2\pi)$ and hence deduce that
$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+$----------.</td><td>07</td></tr>
<tr><td></td><td>c</td><td>Express y as Fourier Series upto second harmonic given:
<table><tr><td>x</td><td>0</td><td>60</td><td>120</td><td>180</td><td>240</td><td>300</td></tr><tr><td>y</td><td>4</td><td>3</td><td>2</td><td>4</td><td>5</td><td>6</td></tr></table></td><td>07</td></tr>
<tr><td colspan="4">OR</td></tr>
<tr><td>4</td><td>a</td><td>Find the Half range sine series of $\pi x-x^2$ in the interval $(0,\pi)$.</td><td>06</td></tr>
<tr><td></td><td>b</td><td>Obtain the Fourier expansion of the function $f(x)=2x-x^2$ in the interval(0,3).</td><td>07</td></tr>
<tr><td></td><td>c</td><td>Obtain the Fourier expansion of y up to first harmonic given:
<table><tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>y</td><td>9</td><td>18</td><td>24</td><td>28</td><td>26</td><td>20</td></tr></table></td><td>07</td></tr>
<tr><td colspan="4">Module - 3</td></tr>
<tr><td>5</td><td>a</td><td>Find the Fourier sine transform of $e^{-|x|}$ and hence evaluate $\int_0^\infty\frac{x\sin mx}{1+x^2}dx, m>0$</td><td>06</td></tr>
</table>

	b	Find the infinite Fourier cosine transform of $f(x) = e^{-\alpha x}, \alpha > 0$.	07
	c	Find the inverse Z- Transform of $\frac{4z^2-2z}{z^3-5z^2+8z-4}$	07
OR			
6	**a**	Find the Fourier transform of the function $f(x) = \begin{cases} 1 \; for \; \|x\| \leq a \\ 0 \; for \; \|x\| > a \end{cases}$. Hence evaluate $\int_0^\infty \frac{sinx}{x} dx$.	06
	b	Obtain the Z- Transform of $cosn\theta \; and \; sinn\theta$.	07
	c	Solve using Z- Transform $y_{n+2} - 4y_n = 0$ given that $y_0 = 0, y_1 = 2$.	07
Module - 4			
7	**a**	Solve $u_{xx} + u_{yy} = 0$ fo the following square mesh with boundary values as shown in the following figure  	06
	b	Solve numerically $u_{xx} = 0.0625 \; u_{tt}$ Subject to the conditions $u(0,t) = 0 = u(5,t)$, $u(x,0) = x^2(x-5)$ by taking $h = 1 \; for \; 0 \leq t \leq 1$.	07
	c	Find the numerical solution of the parabolic equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ when $u(0,t) = 0 = u(4,t)$ and $u(x,0) = x(4-x)$ by taking $h = 1$ find the values up to t = 5.	07
OR			
8	**a**	Solve numerically the equation $u_t = u_{xx}$ Subject to the conditions $u(0,t) = 0 = u(1,t), t \geq 0$ and $u(x,0) = sin\pi x, 0 \leq x \leq 1$. Carryout computations for two levels taking h = 1/3 and k = 1/36.	06
	b	Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ Subject to $u(0,t) = 0 = u(4,t), \; u_t(x,0) = 0$ and $u(x,0) = x(4-x)$ by taking $h = 1, k = 0.5$ up to four steps	07
	c	Solve the elliptic equation $u_{xx} + u_{yy} = 0$ fo the following square mesh with boundary values as shown. Find the first iterative values of $u_i \; (i = 1 \; to \; 9)$ to the nearest integer.  	07

<table>
<tr><th colspan="4">Module - 5</th></tr>
<tr><td>9</td><td>a</td><td>Using Runge - Kutta method, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2, for\ x = 0.2$, correct to four decimal places, using initial conditions $y(0) = 1, y'(0) = 0$</td><td>06</td></tr>
<tr><td></td><td>b</td><td>Derive Euler’s equation in the standard form $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$</td><td>07</td></tr>
<tr><td></td><td>c</td><td>Find the extremal of the functional $\int_{x_1}^{x_2}\left(y^2 + y'^2 + 2ye^x\right)dx.$</td><td>07</td></tr>
<tr><th colspan="4">OR</th></tr>
<tr><td>10</td><td>a</td><td>Given the differential equation $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ and following table of initial values
<table>
<tr><td>x</td><td>1</td><td>1.1</td><td>1.2</td><td>1.3</td></tr>
<tr><td>y</td><td>2</td><td>2.2156</td><td>2.4649</td><td>2.7514</td></tr>
<tr><td>y'</td><td>2</td><td>2.3178</td><td>2.6725</td><td>3.0657</td></tr>
</table>
Compute y(1.4) by applying Milne’s Predictor – Corrector formula.</td><td>06</td></tr>
<tr><td></td><td>b</td><td>Prove that geodesics of a plane surface are straight lines.</td><td>07</td></tr>
<tr><td></td><td>c</td><td>Find the curves on which the functional $\int_0^1\left[(y')^2 + 12xy\right]dx$ with $y(0) = 0$ and $y(1) = 1$ can be extremised.</td><td>07</td></tr>
</table>

Q.No.	Solution and Scheme	Marks
1 a)	Evaluate (i) $L\left\{\frac{\cos 2t - \cos 3t}{t}\right\}$	
	(ii) $L\{t^2 e^{-3t} \sin 2t\}$	
(i)	$f(t) = \frac{\cos 2t - \cos 3t}{t}$	
	$L\{f(t)\} = \int_s^\infty L\{\cos 2t - \cos 3t\}\, ds$	
	$= \int_s^\infty \left\{\frac{s}{s^2+4} - \frac{s}{s^2+9}\right\} ds$	1m
	$= \left[\frac{1}{2}\log(s^2+4) - \frac{1}{2}\log(s^2+9)\right]_s^\infty$	
	$= \left[\frac{1}{2}\left\{\log(s^2+4) - \log(s^2+9)\right\}\right]_s^\infty$	
	$= \left[\frac{1}{2}\log\frac{s^2+4}{s^2+9}\right]_s^\infty$	
	$= \left[\log\sqrt{\frac{s^2+4}{s^2+9}}\right]_s^\infty$	1m
	$= \log 1 - \log\sqrt{s^2+4/s^2+9}$	
	$= 0 - \log\sqrt{\frac{s^2+4}{s^2+9}}$	
	$L\{f(t)\} = \log\sqrt{\frac{s^2+9}{s^2+4}}$	1m

Q.No.	Solution and Scheme	Marks
ii)	$L(t^2 \sin 2t) = (-1)^2 \frac{d^2}{ds^2} L(\sin 2t)$	
	$= \frac{d}{ds} \frac{d}{ds}\left[\frac{2}{s^2+4}\right]$	
	$= \frac{d}{ds}\left[-\frac{4s}{(s^2+4)^2}\right]$	1m
	$= \frac{(s^2+4)^2(-4) - (-4s)2(s^2+4)\cdot 2s}{(s^2+4)^4}$	
	$= \frac{4(s^2+4)\left[-(s^2+4)+4s^2\right]}{(s^2+4)^4}$	1m
	$= \frac{4(3s^2-4)}{(s^2+4)^3}$	
	$L\{e^{-3t} t^2 \sin 2t\} = \frac{4\left[3(s+3)^2-4\right]}{\left[(s+3)^2+4\right]^3}$	1m
	$s \to s+3$	

Q.No.	Solution and Scheme	Marks
1b)	$f(t) = \begin{cases} t & 0 \le t \le a \\ 2a-t & a \le t \le 2a \end{cases}$	
	$T = 2a$ we have	
	$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$	1m
	$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt\right]$	
	$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} \cdot t\,dt + \int_a^{2a} (2a-t)\, e^{-st} f(t)\,dt\right]$	1m
	$= \frac{1}{1-e^{-2as}} \left[\left\{t \frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{(-s)^2}\right\}_a^a + \right.$	
	$\left.\left\{(2a-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{(-s)^2}\right\}_a^{2a}\right]$	1m
	$L\{f(t)\} = \frac{1}{1-e^{-2as}}\left[-\frac{1}{s}(ae^{-as}-0) - \right.$	
	$\frac{1}{s^2}(e^{-as}-1) -\frac{1}{s}(0-ae^{-as})$	
	$\left. + \frac{1}{s^2}(e^{-2as}-e^{-as})\right]$	1m
	$L\{f(t)\} = \frac{1}{(1-e^{-2as})s^2}\left[-e^{-as}+1+e^{-2as}-e^{-as}\right]$	

Q.No.	Solution and Scheme	Marks
	$L\{f(t)\} = \frac{1}{s^2(1-e^{-2as})}(1-2e^{-as}+e^{-2as})$	1m
	$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$	
	$= \frac{(1-e^{-as})}{s^2(1+e^{-as})} = \frac{e^{as/2}-e^{-as/2}}{s^2(e^{as/2}+e^{-as/2})}$	
	$= \frac{1}{s^2}\frac{\cancel{2}\sinh(as/2)}{\cancel{2}\cosh(as/2)}$	
	$L\{f(t)\} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$	2m

Q.No.	Solution and Scheme	Marks
1 c)	$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$, $y(0) = y'(0) = 0$ $\Rightarrow y''(t) + 4y'(t) + 4y(t) = e^{-t}$ Taking L·T on both sides $L\{y''(t)\} + 4L\{y'(t)\} + L\{y(t)\}\cdot 4 = L\{e^{-t}\}$ $\left[s^2L\{y(t)\} - s\overset{0}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}}\right] +$ $4\left[sL\{y(t)\} - \overset{0}{\cancel{y(0)}}\right] + 4L\{y(t)\} = \frac{1}{s+1}$ $L\{y(t)\}\left[s^2+4s+4\right] = \frac{1}{s+1}$ $L\{y(t)\} = \frac{1}{(s+1)(s+2)^2}$ $y(t) = L^{-1}\left\{\frac{1}{(s+1)(s+2)^2}\right\}$ consider $\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$ Put $s = -1 \Rightarrow A = 1$ Put $s = -2 \Rightarrow C = -1$ Put $s = 0 \Rightarrow B = -1$	1m 1m

Q.No.	Solution and Scheme	Marks
	Hence $\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} + \frac{(-1)}{s+2} + \frac{(-1)}{(s+2)^2}$	②m
	$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)}2.$	
	$L^{-1}\left\{\frac{1}{(s+1)(s+2)^2}\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\} - L^{-1}\left\{\frac{1}{(s+2)^2}\right\}$	
	$y(t) = e^{-t} - e^{-2t} - e^{-2t} \cdot L^{-1}\left\{\frac{1}{s^2}\right\}$	
	$y(t) = e^{-t} - e^{-2t} - e^{-2t} \cdot t$	
	$y(t) = e^{-t} - e^{-2t}\left[1+t\right]$	③m

Q.No.	Solution and Scheme	Marks
2.a	$L^{-1}\left\{\frac{4s+5}{(s+1)^2(s+2)}\right\}$	
	Let $\frac{4s+5}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$	1m
	Put $s = -1 \Rightarrow B = 1$	
	Put $s = -2 \Rightarrow C = -3$	
	Put $s = 0 \Rightarrow A = 3$	
	$\Rightarrow \frac{4s+5}{(s+1)^2(s+2)} = \frac{3}{s+1} + \frac{1}{(s+1)^2} + \frac{(-3)}{s+2}$	3m
	$L^{-1}\left\{\frac{4s+5}{(s+1)^2(s+2)}\right\} = 3L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$	
	$-3L^{-1}\left\{\frac{1}{s+2}\right\}$	1m
	$= 3e^{-t} + e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\} - 3e^{-2t}$	
	$L^{-1}\left\{\frac{4s+5}{(s+1)^2(s+2)}\right\} = 3e^{-t} + e^{-t} \cdot t - 3e^{-2t}$	
	$= e^{-t}(3+t) - 3e^{-2t}.$	1m

Q.No.	Solution and Scheme	Marks

2 b) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ by using convolution thm

$f(t) = L^{-1}\{\bar{f}(s)\}$ $\qquad$ $g(t) = L^{-1}\{\bar{g}(s)\}$

$\bar{f}(s) = \frac{1}{s^2+a^2}$ $\qquad$ $\bar{g}(s) = \frac{s}{s^2+a^2}.$ —— ① m

$f(t) = \frac{\sin at}{a}$ $\qquad$ $g(t) = \cos at$

we have convolution theorem

$$L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$ —— 1m

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_{u=0}^{t} \frac{\sin au}{u} \cos(at-au)\, du$$ —— 1m

$$= \frac{1}{2a}\int_{u=0}^{t} \{\sin(au+at-au) + \sin(au-at+au)\}\, du$$ —— 1m

$$= \frac{1}{2a}\int_{u=0}^{t} [\sin at + \sin(2au-at)]\, du$$

$$= \frac{1}{2a}\left[\sin at\,(u)_0^t - \left(\frac{\cos[2au-at]}{2a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[\sin at\,(t-0) - \frac{1}{2a}[\cancel{\cos at} - \cancel{\cos at}]\right]$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t \sin at}{2a}.$$ —— 3m

Q.No.	Solution and Scheme	Marks
2 c)	$f(t) = \begin{cases} \sin t & 0 \le t < \pi \\ \sin 2t & \pi \le t < 2\pi \\ \sin 3t & t > 2\pi \end{cases}$	
	The given f(t) can be written in the following form by a std property	
	$f(t) = \sin t + [\sin 2t - \sin t]\, u(t-\pi)$	
	$+ [\sin 3t - \sin 2t]\, u(t-2\pi)$	1m
	$L\{f(t)\} = L\{\sin t\} +$	
	$L\{[\sin 2t - \sin t]\ u(t-\pi)\} +$	
	$L\{[\sin 3t - \sin 2t]\, u(t-2\pi)\}$ ①	1m
	consider $L\{[\sin 2t - \sin t]\, u(t-\pi)\}$	
	$F(t-\pi) = \sin 2t - \sin t$	
	$F(t) = \sin 2(t+\pi) - \sin(t+\pi)$	
	$F(t) = \sin(2t+2\pi) - \sin(\pi+t)$	
	$F(t) = \sin 2t + \sin t.$	
	$\bar{F}(s) = L\{F(t)\} = L\{\sin 2t\} + L\{\sin t\}$	
	$\bar{F}(s) = \frac{2}{s^2+4} + \frac{1}{s^2+1}$	2m

But $L\{F(t-\pi)\,u(t-\pi)\} = e^{-as}\,\bar{F}(s)$

ie $L\{[\sin 2t - \sin t]\,u(t-\pi)\}$

$$= e^{-as}\left\{\frac{2}{s^2+4} + \frac{1}{s^2+1}\right\}$$

Also $G(t-2\pi) = \sin 3t - \sin 2t$

$G(t) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$

$G(t) = \sin 3t - \sin 2t$

$$\bar{G}(s) = \frac{3}{s^2+9} - \frac{2}{s^2+4}$$ 1m

But $L\{G(t-2\pi)\,u(t-2\pi)\} = e^{-2as}\,\bar{G}(s)$

$L\{[\sin 3t - \sin 2t]\,u(t-2\pi)] =$

$$e^{-2as}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$$

Using these in Eqn (1)

$$L\{f(t)\} = \frac{1}{s^2+1} + e^{-as}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right]$$

$$+ e^{-2as}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$$ 2m

Q.No.	Solution and Scheme	Marks
3 (a)	Let $f(x) = \lvert x \rvert$ in $(-\pi, \pi)$	
	$f(x) = \lvert x \rvert$ in $-\pi \le x \le \pi$ means that the function must be positive in the given interval which consists of negative values and positive values. Hence the given $f(x)$ may be split into the form	
	$f(x) = \begin{cases} -x & \text{in } -\pi \le x \le 0 \\ x & \text{in } 0 \le x \le \pi \end{cases}$	
	Period $T = b - a = \pi - (-\pi) = 2\pi$	
	The F.S of $f(x)$ having period 2π is given by	
	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$	(1m) ①
	We shall check for even or odd nature	
	$f(-x) = \lvert -x \rvert = \lvert x \rvert = f(x)$	
	$\Rightarrow f(x)$ is even consequently	
	$b_n = 0$ where	(1m)
	$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$	
	$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$	

Q.No.	Solution and Scheme	Marks

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \frac{\cancel{2}}{\pi}\cdot\frac{1}{\cancel{2}}\left[\pi^2-0\right] = \frac{1}{\pi}\cdot\pi^2 = \pi$$

$$a_0 = \pi \quad \Rightarrow \quad \frac{a_0}{2} = \frac{\pi}{2}.$$ 2m

Applying Bernoulli's rule to find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx.\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[x\,\frac{\sin nx}{n} - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

since $\sin n\pi = \sin 0 = 0$

$$\Rightarrow a_n = \frac{1}{n^2}\cdot\frac{2}{\pi}\left[\cos nx\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi n^2}\left[\cos \pi n - \cos 0\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$a_n = -\frac{2}{\pi n^2}\left[1-(-1)^n\right]$$

Substituting in Eqn ①

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{-2}{\pi n^2}\left\{1-(-1)^n\right\}\cos nx.$$ 2m

Q.No. **Solution and Scheme** **Marks**

3(b) $f(x) = \frac{(\pi - x)^2}{4}$ in $(0, 2\pi)$

Period $T = b - a = 2\pi - 0 = 2\pi$.

The F.S of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ①$$ 1m

we shall check for even or odd nature

$$f(2\pi - x) = \frac{[\pi - (2\pi - x)]^2}{4} = \frac{(\pi - x)^2}{4}$$

$f(2\pi - x) = f(x) \Rightarrow f(x)$ is even function in $(0, 2\pi)$ hence $b_n = 0$ 1m

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{(\pi - x)^2}{4}\,dx$$

$$a_0 = \frac{1}{2\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{\pi} = -\frac{1}{6\pi}\left[0 - \pi^3\right] = \frac{\pi^2}{6}$$

$$a_0 = \frac{\pi^2}{6} \Rightarrow \frac{a_0}{2} = \frac{\pi^2}{12}.$$ 1m

$$a_n = \frac{2}{\pi}\int_0^{\pi} \frac{(\pi - x)^2}{4} \cos nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)^2 \cdot \frac{\sin nx}{n} - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

Q.No.	Solution and Scheme	Marks
	$a_n = -\frac{1}{\pi n^2}\left[(\pi - x)\cos nx\right]_0^{\pi}$	
	$a_n = \frac{-1}{\pi n^2}\left[0 - \pi(1)\right] = \frac{-1}{\cancel{\pi} n^2}(-\cancel{\pi})$	
	$a_n = \frac{1}{n^2}$ —	②m
	Thus required F.S is	
	$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ — ②	
	Put $x = \pi$ in F.S	
	$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi$	
	$\frac{-\pi^2}{12} = \frac{1}{n^2} \sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n$	
	$\frac{-\pi^2}{12} = \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \right\}$	
	$\frac{-\pi^2}{12} = \frac{(-1)^1}{(1)^2} + \frac{(-1)^2}{2^2} + \frac{(-1)^3}{3^2} + \frac{(-1)^4}{4^2} + \cdots$	
	$\frac{-\pi^2}{12} = -\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$	
	$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ —	②m

Q.No.	Solution and Scheme	Marks

3C) Express y as a F.S upto second harmonics

x	y	y cosx	y cos2x	y sinx	y sin2x
0	4	4	4	0	0
60	3	1.5	−1.5	2.598	2.598
120	2	−1	−1	1.732	−1.732
180	4	−4	4	0	0
240	5	−2.5	−2.5	−4.3	4.3
300	6	3	−3	−5.196	−5.196
Σy = 24		Σycosx = 1	Σycos2x = 0	Σysinx = −5.166	Σysin2x = −0.03

3m

$$a_0 = \frac{2}{N}\Sigma y = \frac{\cancel{2}}{\cancel{6}_{\cancel{3}}}(\overset{8}{\cancel{24}}) = 8$$

$$\Rightarrow a_0/2 = \frac{8}{2} = 4$$

$$a_1 = \frac{2}{N}\Sigma y\cos x = \frac{2}{8}(1) = \frac{1}{4} = 0.25$$

2m

$$a_2 = \frac{2}{N}\Sigma y\cos 2x = \frac{2}{8}(0) = 0$$

$$b_1 = \frac{2}{N}\Sigma y\sin x = \frac{2}{8}(-5.166) = -1.2915$$

$$b_2 = \frac{2}{N}\Sigma y\sin 2x = \frac{2}{8}(-0.03) = -7.5\times 10^{-3}$$

F.S having period 2π upto second harmonics is given by

$$f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$$

$$f(x) = 4 + [(0.25)\cos x + (-1.2915)\sin x] + [(0)\cos 2x + (-7.5\times 10^{-3})\sin 2x]$$

2m

Q.No.	Solution and Scheme	Marks
4a)	$f(x) = \pi x - x^2$ in $(0, \pi)$	
	Half range sine series is given by	
	$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \, dx$	1m.
	$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$	
	$b_n = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin nx$	
	$b_n = \frac{2}{\pi} \Big[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (0-2)\left(\frac{\cos nx}{n^3}\right) \Big]_0^{\pi}$	1m
	since $\sin 0 = \sin n\pi = 0$	
	$b_n = \frac{2}{\pi} \Big[\frac{1}{n} \{ (\pi x - x^2) \cos nx \}_0^{\pi} + \frac{(-2)}{n^3} \{ \cos nx \}_0^{\pi} \Big]$	2m
	$b_n = \frac{2}{\pi} \Big[\frac{1}{n} [0 - 0] - \frac{2}{n^3} [\cos n\pi - \cos 0] \Big]$	
	$b_n = \frac{2}{\pi} \Big[-\frac{2}{n^3} \{ (-1)^n - 1 \} \Big] = \frac{4}{\pi n^3} [1 - (-1)^n]$	
	$\Rightarrow f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin nx$	2m

Q.No. | **Solution and Scheme** | **Marks**

4b) Let $f(x) = 2x - x^2$ in $(0, 3)$

Period $T = b - a = 3 - 0 = 3$, $2l = 3$

$l = \frac{3}{2}$

F.S of $f(x)$ having period $3/2$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi x}{3}$$ (1m)

①

$$a_0 = \frac{2}{3} \int_0^3 f(x)\, dx$$

$$a_n = \frac{2}{3} \int_0^3 f(x) \cos \frac{2n\pi x}{3}\, dx$$

$$b_n = \frac{2}{3} \int_0^3 f(x) \sin \frac{2n\pi x}{3}\, dx$$

$$a_0 = \frac{2}{3} \int_0^3 (2x - x^2)\, dx = \frac{2}{3} \left[\cancel{2} \cdot \frac{x^2}{\cancel{2}} - \frac{x^3}{3} \right]_0^3$$

$$a_0 = \frac{2}{3} \{ (9 - 9) - 0 \} = 0 \Rightarrow \frac{a_0}{2} = 0$$ (1m)

$$a_n = \frac{2}{3} \int_0^3 (2x - x^2) \cos \frac{2n\pi x}{3}\, dx.$$

Applying Bernoulli's rule

$$a_n = \frac{2}{3} \left\{ (2x - x^2) \cdot \frac{\sin 2n\pi x/3}{2n\pi/3} - (2 - 2x) \left[\frac{-\cos 2n\pi x/3}{(2n\pi/3)^2} \right] + (-2) \left[\frac{-\sin 2n\pi x/3}{(2n\pi/3)^3} \right] \right\}_0^3$$

Since $\sin 0 = 0 = \sin 2n\pi$.

$$a_n = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{9}^{3}}{\cancel{4}_{2}\,\pi^2 n^2}\left[(2-2x)\cos\frac{2n\pi x}{3}\right]_0^3$$

$$a_n = \frac{3}{2\pi^2 n^2}\left[-4\cos 2n\pi - 2\cos 0\right]$$

$$a_n = \frac{3}{2\pi^2 n^2}(-4-2) = -\frac{\cancel{18}^{9}}{\cancel{2}\,\pi^2 n^2} = \underline{\frac{-9}{\pi^2 n^2}}$$ (2m)

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2\pi n x}{3}\,dx$$

Applying Bernoulli's rule

$$b_n = \frac{2}{3}\left\{(2x-x^2)\left(\frac{-\cos 2n\pi x/3}{2n\pi/3}\right) - (2-2x)\left(\frac{-\sin 2n\pi x/3}{[2n\pi/3]^2}\right) + (-2)\left(\frac{+\cos 2n\pi x/3}{[2n\pi/3]^3}\right)\right\}_0^3$$

$$b_n = \frac{2}{3}\left\{\frac{-3}{2n\pi}\left[(2x-x^2)\cos 2n\pi x/3\right] - \frac{54}{8n^3\pi^3}\cos 2n\pi x/3\right\}_0^3$$

$$b_n = \frac{2}{3}\left\{\frac{-3}{2n\pi}\left[(-3)\cos 2n\pi - 0\right] - \frac{54}{8n^3\pi^3}\left[\cos 2n\pi - \cos 0\right]\right\}$$

$$b_n = \frac{\cancel{2}}{\cancel{3}}\left\{\frac{-\cancel{3}}{\cancel{2}n\pi}\left[(-3)\right]\right\}$$ $\because \cos 2n\pi = 1$, $\cos 0 = 1$

Q.No.	Solution and Scheme	Marks

$b_n = \frac{3}{n\pi}$ —— (2m)

Substituting in Eqn ①

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{-9}{\pi^2 n^2} \cos \frac{2n\pi x}{3} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin \frac{2n\pi x}{3}$$

$$f(x) = -\frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2n\pi x}{3} + \frac{3}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{3}$$ —— (1m)

4c) Express y upto first harmonics

Here $N = 6$, hence the interval should be $0 \leq x < 6$

length of interval $= b - a = 6 - 0 = 6$

$2l = 6 \Rightarrow l = 3$. The F.S of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1} b_n \sin \frac{n\pi x}{l}$$

since $l = 3$, the series containing the first harmonic is

$$y = f(x) = \frac{a_0}{2} + a_1 \cos \frac{\pi x}{3} + b_1 \sin \frac{\pi x}{3}.$$

$$\theta = \frac{\pi x}{l} \Rightarrow \theta = \frac{\pi x}{3}$$

$$\Rightarrow y = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$$ —— (1m)

$N = 6 \Rightarrow 2/N = 2/6 = 1/3$

Q.No. **Solution and Scheme** **Marks**

x	$\theta = \pi x/3$	y	$y\cos\theta$	$y\sin\theta$
0	0	9	9	0
1	60°	18	9	15.588
2	120°	24	-12	20.784
3	180°	28	-28	0
4	240°	26	-13	-22.516
5	300°	20	10	-17.32
		$\Sigma y =$ 125	$\Sigma y\cos\theta =$ -25	$\Sigma y\sin\theta =$ -3.464

(3m)

$$a_0 = \frac{2}{N}\Sigma y = \frac{1}{3}(125) = 41.67$$

$$\frac{a_0}{2} = \frac{41.67}{2} = 20.835$$

$$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{1}{3}(-25) = -8.333$$

$$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{1}{3}(-3.464)$$ — (2m)

$$b_1 = -1.155$$

F.S expansion of y upto first harmonic is given by

$$f(x) = 20.835 + \{(-8.333)\cos x + (-1.155)\sin x\}.$$ — (1m)

Q.No.	Solution and Scheme	Marks
5 a)	$f(x) = e^{-\lvert x \rvert}$	
	Fourier sine transform is given by	
	$F_s(u) = \int_0^\infty f(x) \sin ux \, dx.$	(1m)
	$F_s(u) = \int_0^\infty e^{-\lvert x \rvert} \sin ux \, dx$	
	$= \int_0^\infty e^{-x} \sin ux \, dx \quad \because \lvert x \rvert = x, x > 0$	
	$F_s(u) = \left[\frac{e^{-x}}{(-1)^2 + u^2} \{(-1) \sin ux - u \cos ux\} \right]_0^\infty$	
	$F_s(u) = \frac{1}{1+u^2} \left[e^{-x} (-\sin ux - u \cos ux) \right]_0^\infty$	
	$F_s(u) = \frac{1}{1+u^2} [0 - e^{-0} (0 - u)]$	
	$F_s(u) = \frac{u}{1+u^2}$	(2m)
	By inverse Fourier sine transform	
	$\frac{2}{\pi} \int_0^\infty F_s(u) \sin ux \, du = f(x)$	(1m)
	$\int_0^\infty F_s(u) \sin ux \, du = \frac{\pi}{2} f(x)$	
	Put $x = m$ where $m > 0$ we have	
	$f(x) = e^{-\lvert m \rvert} = e^{-m}, \quad m > 0$	

Q.No.	Solution and Scheme	Marks

$$\Rightarrow \int_0^\infty \frac{u}{1+u^2} \sin um \, du = \frac{\pi}{2} e^{-m}$$

Thus by changing the variable u to x.

$$\int_0^\infty \frac{x \sin mx}{1+x^2} \, dx = \frac{\pi}{2} e^{-m}$$ — (2m)

5b) $f(x) = e^{-\alpha x}$

Infinite fourier cosine transform of f(x) is given by

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$ — (1m)

$$F_c(u) = \int_0^\infty e^{-\alpha x} \cos ux \, dx$$ — (1m)

$$= \left[\frac{e^{-\alpha x}}{(-\alpha)^2 + u^2} (-\alpha \cos ux + u \sin ux) \right]_0^\infty$$

$$= \frac{1}{\alpha^2+u^2} \left[e^{-\alpha x} (-\alpha \cos ux + u \sin ux) \right]_0^\infty$$

$$= \frac{1}{\alpha^2+u^2} \left[0 - e^{-0} (-\alpha + 0) \right]$$ — (4m)

$$F_c(u) = \frac{\alpha}{\alpha^2+u^2}.$$ — (1m)

Q.No.	Solution and Scheme	Marks

5 c)

$$z^{-1}\left\{\frac{4z^2-2z}{z^3-5z^2+8z-4}\right\}$$

$$u(z) = \frac{4z^2-2z}{z^3-5z^2+8z-4}$$

$$u(z) = \frac{z(4z-2)}{z^3-5z^2+8z-4}$$

$$\frac{u(z)}{z} = \frac{4z-2}{z^3-5z^2+8z-4}$$ —— (1m)

we shall factorize the DR first.

$z^3-5z^2+8z-4 = (z-1)(z-2)^2$. —— (1m)

$$\frac{u(z)}{z} = \frac{4z-2}{(z-1)(z-2)^2}$$

consider $\frac{4z-2}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$

$4z-2 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$

Put $z=2$. $\Rightarrow$ $C = 6$

$z = 1$ $\Rightarrow$ $A = 2$.

Equating coefficient of z^2 both sides

$0 = A + B \Rightarrow B = -A \quad B = -2.$

$$\frac{4z-2}{(z-1)(z-2)^2} = \frac{2}{z-1} + \frac{(-2)}{z-2} + \frac{6}{(z-2)^2}.$$ —— 4m

Q.No.	Solution and Scheme	Marks

$$\frac{4z-2}{(z-1)(z-2)^2} = \frac{2}{z-1} - \frac{2}{z-2} + \frac{6}{(z-2)^2}$$

$$\frac{u(z)}{z} = \frac{2}{z-1} - \frac{2}{z-2} + \frac{6}{(z-2)^2}$$

$$u(z) = 2\,\frac{z}{z-1} - 2\,\frac{z}{z-2} + 3\cdot\frac{2z}{(z-2)^2}.$$

—(2m)

Taking inverse Z-T bothsides

$$z^{-1}\left\{u(z)\right\} = 2z^{-1}\left\{\frac{z}{z-1}\right\} - 2z^{-1}\left\{\frac{z}{z-2}\right\} + 3z^{-1}\left\{\frac{2z}{(z-2)^2}\right\}$$

$$u_n = 2\cdot(1)^n - 2\cdot(2)^n + 3\cdot n\cdot 2^n$$

$$u_n = 2 - 2^{n+1} + 3n\,2^n.$$

—(1m)

Q.No.	Solution and Scheme	Marks
5 a)	$f(x) = \begin{cases} 1, & \lvert x \rvert \leq a \\ 0, & \lvert x \rvert > a \end{cases}$	
	Fourier transform of $f(x)$ is given by	
	$F(u) = \int_{x=-\infty}^{\infty} f(x)\, e^{iux}\, dx$ ———	(1m)
	We write $f(x)$ as	
	$f(x) = \begin{cases} 1 & -a \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$	
	$F(u) = \int_{x=-a}^{a} 1\, e^{iux}\, dx = \left[\frac{e^{iux}}{iu}\right]_{-a}^{a}$	
	$F(u) = \frac{1}{iu}\left\{ e^{iua} - e^{-iua} \right\}$	
	$F(u) = \frac{1}{iu}\left\{ (\cos au + i\sin au) - (\cos au - i\sin au) \right\}$	
	$F(u) = \frac{1}{\cancel{i}u}\, 2\cancel{i}\, \sin au = \frac{2\sin au}{u}$	
	$F(u) = \frac{2\sin au}{u}$ ———	(2m)

Q.No.	Solution and Scheme	Marks
	let us evaluate $\int_0^\infty \frac{\sin x}{x} dx$	
	we have obtained $F(u) = \frac{2\sin au}{u}$	
	Inverse F.T is $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux} du = f(x)$	
	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin au}{u} e^{-iux} du.$	
	$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u} e^{-iux} du$	(1m)
	Now let us put $x = 0$	
	$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u} e^{-0} du$	
	$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u} du$ ∵ $f(0) = 1.$	
	$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin au}{u} du = \pi.$	
	$2\int_0^{\infty} \frac{\sin au}{u} du = \pi$ ∵ $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ for Even fun	
	$\Rightarrow \int_0^{\infty} \frac{\sin au}{u} du = \frac{\pi}{2}$ put $a = 1$ & $u = x$	
	$\Rightarrow \int_0^{\infty} \frac{\sin x}{x} dx = \pi/2$ #	(2m)

5 b) obtain $Z(\cos n\theta)$ & $Z(\sin n\theta)$

we know that $e^{in\theta} = \cos n\theta + i\sin n\theta$

$e^{in\theta} = (e^{i\theta})^n = a^n$ where $a = e^{i\theta}$ —(1m)

w.k.t $Z(a^n) = \frac{z}{z-a}$

$$Z(a^n) = \frac{z}{z-e^{i\theta}}$$

$$Z(a^n) = \frac{z}{z-e^{i\theta}} \quad \frac{z-e^{-i\theta}}{z-e^{-i\theta}}$$ —(1m)

$$= \frac{z[z-e^{-i\theta}]}{z^2-(e^{i\theta}-e^{-i\theta})+1}$$ —(1m)

$$= \frac{z[z-(\cos\theta-i\sin\theta)]}{z^2-2z\cos\theta+1}$$

$\because e^{i\theta}-e^{-i\theta} = 2\cos\theta$

$$Z(a^n) = \frac{z[(z-\cos\theta)+i\sin\theta]}{z^2-2z\cos\theta+1}$$

$$Z[\cos n\theta+i\sin n\theta] = \left\{\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right\} + i\left\{\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right\}$$ —(1m)

Q.No. **Solution and Scheme** **Marks**

separating the real and imaginary parts bothsides we get

$$Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$ (1m)

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1.}$$ (1m)

5 c) $y_{n+2} - 4y_n = 0 \quad y_0 = 0 \quad y_1 = 2.$

Taking Z-T on bothsides

$$Z\{y_{n+2}\} - 4Z\{y_n\} = Z(0)$$

$$z^2\{y(z) - y_0 - \frac{y_1}{z}\} - 4y(z) = 0$$ (1m)

$$z^2\{y(z) - 0 - \frac{1}{z}(2)\} - 4y(z) = 0$$

$$y(z)[z^2-4] - \frac{2}{\cancel{z}}\cdot z^{\cancel{2}} = 0$$

$$(z^2-4)\;y(z) = \frac{2z}{(z^2-4)}$$

$$y(z) = \frac{2z}{z^2-4} \Rightarrow \frac{y(z)}{z} = \frac{2}{z^2-4}$$ (1m)

Consider $\frac{2}{z^2-4} = \frac{2}{(z+2)(z-2)}$

$$\frac{2}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$$

Q.No. | **Solution and Scheme** | **Marks**

$2 = A(z+2) + B(z-2)$

Put $z = 2 \Rightarrow A = 1/2$

$z = -2 \Rightarrow B = -1/2$.

$$\frac{2}{z^2-4} = \frac{A}{z-2} + \frac{B}{z+2} = \frac{(1/2)}{z-2} + \frac{(-1/2)}{z+2}$$

$$\frac{2}{z^2-4} = \frac{1}{2}\frac{1}{z-2} - \frac{1}{2}\frac{1}{z+2}$$ —(2m)

$$\frac{y(z)}{z} = \frac{1}{2}\frac{1}{z-2} - \frac{1}{2}\frac{1}{z+2}$$

$$y(z) = \frac{1}{2}\frac{z}{z-2} - \frac{1}{2}\frac{z}{z+2}$$ —(1m)

Taking inverse Z-T both sides

$$z^{-1}\left\{y(z)\right\} = \frac{1}{2} z^{-1}\left\{\frac{z}{z-2}\right\} - \frac{1}{2} z^{-1}\left\{\frac{z}{z+2}\right\}$$

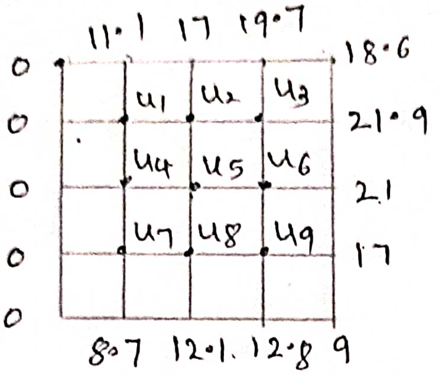
$$y_n = \frac{1}{2}(2)^n - \frac{1}{2}(-2)^n$$

$y_n = 2^{n-1} + (-2)^{n-1}$ is required soln. —(2m)

$$\because z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$$

Q.No.	Solution and Scheme	Marks

7 a) Solve $u_{xx}+u_{yy}=0$ for the following square mesh with boundary values as shown in the following figure



S·F → std 5 pt formula

D·F → Diagonal 5 pt formula

Sol? u_5 is located at the centre of the region

$u_5 = \frac{1}{4}(0+21+17+21\cdot1) = 12\cdot525$ (S·F)

Next we shall compute u_7, u_9, u_1, u_3 by applying D·F

$u_7 = \frac{1}{4}(0+12\cdot525+0+12\cdot1) = 6\cdot15625$

$u_9 = \frac{1}{4}(12\cdot1+21+12\cdot525+9) = 13\cdot65625$ — (2m)

$u_1 = \frac{1}{4}(0+17+0+12\cdot525) = 7\cdot38125$

$u_3 = \frac{1}{4}(12\cdot525+18\cdot6+17+21) = 17\cdot28125$

Finally we shall compute u_2, u_4, u_6 & u_8 by S·F

$u_2 = \frac{1}{4}(7\cdot38125+17\cdot28125+17+12\cdot525)$

$u_2 = 13\cdot546875$

$u_4 = \frac{1}{4}(0+12\cdot525+7\cdot38125+6\cdot15625)$ — (3m)

$u_4 = 6\cdot515625$

$u_6 = \frac{1}{4}(12\cdot525+21+17\cdot28125+13\cdot65625)$

$u_6 = 16\cdot115625$

$u_8 = \frac{1}{4}(6\cdot15625+13\cdot65625+12\cdot525+12\cdot1)$

$u_8 = 11\cdot109375$

Q.No.	Solution and Scheme	Marks
	Thus the required values of (x, y) at the interior mesh points correct to two decimal places are as follows $u_1 = 7.38$ $u_2 = 13.55$ $u_3 = 17.28$ $u_4 = 6.52$ $u_5 = 12.53$ $u_6 = 16.12$ $u_7 = 6.16$ $u_8 = 11.11$ $u_9 = 13.66$	1m

7 b) solve numerically $u_{xx} = 0.0625 u_{tt}$ subject to the condition $u(0,t) = 0$ $u(5,t) = 0$, $u(x,0) = x^2(x-5)$ & $u_t(x,0) = 0$ by taking $h = 1$ for $0 \le t \le 1$

Sol? The wave Eqn in std form is $c^2 u_{xx} = u_{tt}$ hence given Eqn be put in the form $\frac{1}{0.0625} u_{xx} = u_{tt}$ or

$16 u_{xx} = u_{tt}$ where $c^2 = 16 \Rightarrow c = 4$

since $h = 1$ we have $k = \frac{h}{c} = \frac{1}{4} = 0.25$

step size of x : $h = 1$ where $0 \le x \le 5$

step size of t : $k = 0.25$ where $0 \le t \le 1$ (1m)

values of x are 0, 1, 2, 3, 4, 5

values of t are 0, 0.25, 0.5, 0.75, 1

we have the following table of initial table. The value of 1st and last column are zero since $u(0,t) = 0 = u(5,t)$

x t		x_0 0	x_1 1	x_2 2	x_3 3	x_4 4	x_5 5
t_0	0	$u_{0,0} = 0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$	$u_{5,0} = 0$
t_1	0.25	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	$u_{5,1} = 0$
t_2	0.5	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$	$u_{5,2} = 0$
t_3	0.75	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$	$u_{5,3} = 0$
t_4	1	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$	$u_{5,4} = 0$

Now consider $u(x,0) = x^2(x-5)$ (1m)

$u(1,0) = 1^2(1-5) = -4$ $u(2,0) = 2^2(2-5) = -12$.

$u(3,0) = 3^2(3-5) = -18$ $u(4,0) = 4^2(4-5) = -16$

Q.No.	Solution and Scheme	Marks

Next consider $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$

$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0-12] = -6$

$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[-4-18] = -11$

$u_{3,1} = \frac{1}{2}[u_{2,0}) + u_{4,0}] = \frac{1}{2}[-12-16] = -14$

$u_{4,1} = \frac{1}{2}[u_{3,0} + u_{5,0}] = \frac{1}{2}[-18+0] = -9$ — (1m)

we consider Explicit formula to find the remaining values in the table

$u_{i\,j+1} = u_{i-1\,j} + u_{i+1,\,j} - u_{i\,j-1}$

$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 - 11 + 4 = -7$

$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0}' = -6 - 14 + 12 = -8$ — (1m)

$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = -11 - 9 + 18 = -2.$

$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = -14 + 0 + 16 = 2.$

$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 - 8 + 6 = -2.$

$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = -7 - 2 + 11 = 2$

$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = -8 + 2 + 14 = 8$ — (1m)

$u_{4,3} = u_{3,2} + u_{5,2} - u_{4,1} = -2 + 0 + 9 = 7$

$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 + 2 + 7 = 9$

$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -2 + 8 + 8 = 14$

$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = 2 + 7 + 2 = 11$ — (1m)

$u_{4,4} = u_{3,3} + u_{5,3} - u_{4,2} = 8 + 0 - 2 = 6$

Thus the required value of u_{ij} are tabulated

t \ x	0	1	2	3	4	5
0	0	-4	-12	-18	-16	0
0.25	0	-6	-11	-14	-9	0
0.5	0	-7	-8	-2	2	0
0.75	0	-2	2	8	7	0
1	0	9	14	11	6	0

— (1m)

7c) Find the numerical solution of parabolic Eqn $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$ when $u(0,t) = 0 = u(4,t)$ and $u(x,0) = x(4-x)$ by taking $h=1$ find the values upto $t = 5$

Sol: The std form of one dimensional heat Eqn is $u_t = c^2 u_{xx}$ and given Eqn can be put in the form $u_t = \frac{1}{2} u_{xx} \Rightarrow c^2 = \frac{1}{2}$

since $h = 1 \Rightarrow k = h^2/2c^2 = \frac{1}{2 \cdot \frac{1}{2}} = 1$ (1m)

The values of x in $0 \leq x \leq 4$ with $h = 1$ are 0, 1, 2, 3, 4 and the values of t with $k = 1$ are 0, 1, 2, 3, 4, 5. given that $u(0,t) = 0 = u(4,t) \Rightarrow$ The values in the first and last column are zero.

x / t		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	$u_{0,0} = 0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0} = 0$
t_1	1	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1} = 0$
t_2	2	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2} = 0$
t_3	3	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3} = 0$
t_4	4	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4} = 0$
t_5	5	$u_{0,5} = 0$	$u_{1,5}$	$u_{2,5}$	$u_{3,5}$	$u_{4,5} = 0$

Consider the initial condition $u(x,0) = x(4-x)$

$u(1,0) = 1(4-1) = 3$, $u(2,0) = 2(4-2) = 4$ (1m)

$u(3,0) = 3(4-3) = 3$

$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$ —— ①

Inperticular $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$ (2m)

$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0+4] = 2$

$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[3+3] = 3$

$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = \frac{1}{2}[4+0] = 2.$

Again from Eqn ① $u_{i,2} = \frac{1}{2}[u_{i-1,1} + u_{i+1,1}]$

$u_{1,2} = \frac{1}{2}[u_{0,1} + u_{2,1}] = \frac{1}{2}[0+3] = 1.5$

$u_{2,2} = \frac{1}{2}[u_{1,1} + u_{3,1}] = \frac{1}{2}[2+2] = 2$ (1m)

$u_{3,2} = \frac{1}{2}[u_{2,1} + u_{41}] = \frac{1}{2}[3+0] = 1.5$

Again from eqn ① $u_{i,3} = \frac{1}{2}[u_{i-1,2} + u_{i+1,2}]$

$u_{1,3} = \frac{1}{2}[u_{02} + u_{22}] = \frac{1}{2}[0+2] = 1$

$u_{2,3} = \frac{1}{2}[u_{1,2} + u_{3,2}] = \frac{1}{2}[1.5+1.5] = 1.5$ (1m)

$u_{3,3} = \frac{1}{2}[u_{2,2} + u_{4,2}] = \frac{1}{2}[2+0] = 1$

Also from ① $u_{i,4} = \frac{1}{2}[u_{i-1,3} + u_{i+1,3}]$

$u_{1,4} = \frac{1}{2}[u_{0,3} + u_{2,3}] = \frac{1}{2}[0+1.5] = 0.75$ (1m)

$u_{2,4} = \frac{1}{2}[u_{1,3} + u_{3,3}] = \frac{1}{2}[1+1] = 1$

$u_{3,4} = \frac{1}{2}[u_{2,3} + u_{4,3}] = \frac{1}{2}[1.5+0] = 0.75$

Again from ① $u_{i,5} = \frac{1}{2}[u_{i-1,4} + u_{i+1,4}]$

$u_{1,5} = \frac{1}{2}[u_{0,4} + u_{2,4}] = \frac{1}{2}[0+1] = 0.5$

$u_{2,5} = \frac{1}{2}[u_{1,4} + u_{3,4}] = \frac{1}{2}[0.75+0.75] = 0.75$ (1m)

$u_{3,5} = \frac{1}{2}[u_{2,4} + u_{4,4}] = \frac{1}{2}[1+0] = 0.5$

Thus the required values of u_{ij} are tabulated

$t \backslash x$	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

(1m)

Q.No.	**Solution and Scheme**	**Marks**
8 a)	solve numerically the eqn $u_t = u_{xx}$	
	subject to the conditions $u(0,t) = 0 = u(1,t)$	
	$t \geq 0$ and $u(x,0) = \sin \pi x$, $0 \leq x \leq 1$	
	carryout computations for two levels	
	taking $h = 1/3$ and $k = 1/36$	
Sol:	we have Schmidt explicit formula	
	$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha) u_{i,j} + \alpha u_{i+1,j}$ — ①	
	where $\alpha = k c^2/h^2$ we have $c^2 = 1$, $h = 1/3$	
	$k = 1/36$ ∴ $\alpha = 1/4$	(1m)
	Hence eqn ① becomes	
	$u_{i,j+1} = \frac{1}{4} u_{i-1,j} + \frac{1}{2} u_{i,j} + \frac{1}{4} u_{i+1,j}$	
	$u_{i,j+1} = \frac{1}{4} [u_{i-1,j} + 2u_{i,j} + u_{i+1,j}]$ — ②	(1m)
	since $h = 1/3$ the values of x in $0 \leq x \leq 1$	
	$x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 3/3 = 1$	
	$u(0,t) = 0 \Rightarrow u_{0,0} = u_{0,1} = u_{0,2} = u_{0,3} = 0$	
	$u(1,t) = 0 \Rightarrow u_{3,0} = u_{3,1} = u_{3,2} = u_{3,3} = 0$	(1m)
	Also $u(x,0) = \sin \pi x$, hence we have	
	$u(x_1,0) = u_{1,0} = \sin \pi/3 = 0.866$	(1m)
	$u(x_2,0) = u_{2,0} = \sin 2\pi/3 = 0.866$	
	We shall compute $u_{1,1}$, $u_{2,1}$ (1st level)	
	and $u_{1,2}$, $u_{2,2}$ (2nd level) using Eqn ②	
	$u_{1,1} = \frac{1}{4}[u_{0,0} + 2u_{1,0} + u_{2,0}] = 0.6495$	
	$u_{2,1} = \frac{1}{4}[u_{1,0} + 2u_{2,0} + u_{3,0}] = 0.6495$	(2m)
	$u_{1,2} = \frac{1}{4}[u_{0,1} + 2u_{1,1} + u_{2,1}] = 0.487125$	
	$u_{2,2} = \frac{1}{4}[u_{1,1} + 2u_{2,1} + u_{3,1}] = 0.487125$	

8 b) Solve the wave Eqn $\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}$

subject to $u(0,t) = 0 = u(4,t)$

$u_t(x,0) = 0$ and $u(x,0) = x(4-x)$

by taking h = 1 k = 0.5 upto 4 steps

Ans: step size of x : h = 1

step size of t : k = 0.5

since $0 \leq x \leq 4$, the pts of x are (1m)

0, 1, 2, 3, 4

since k = 0.5, the pts of t are

0, 0.5, 1, 1.5, 2

The values of 1st & last column are zero since $u(0,t) = 0 = u(4,t)$

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	$u_{00} = 0$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0} = 0$
t_1	0.5	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1} = 0$
t_2	1	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2} = 0$
t_3	1.5	$u_{0,3} = 0$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3} = 0$
t_4	2	$u_{0,4} = 0$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4} = 0$

Now consider $u(x,0) = x(4-x)$

$u_{1,0} = 1(4-1) = 3$ $u_{2,0} = 2(4-2) = 4$ (1m)

$u_{3,0} = 3(4-3) = 1$

Next consider $u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$

$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0+4] = 2$ (1m)

$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[3+3] = 3$

$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = \frac{1}{2}[4+0] = 2.$

Now consider explicit formula to find remaining values

$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,1} = 0 + 3 - 3 = 0$

$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 2 + 2 - 4 = 0$ (1m)

$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 3 + 0 - 3 = 0$

$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 + 0 - 2 = -2.$

$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = 0 + 0 - 3 = -3$ (1m)

$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 0 + 0 - 2 = -2.$

$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 - 3 - 0 = -3$

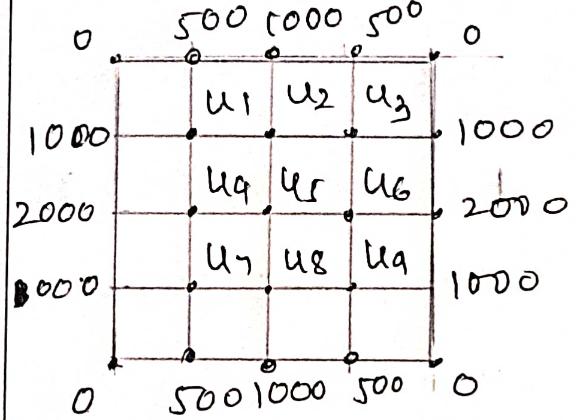
$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -2 - 2 - 0 = -4$ (1m)

$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -3 + 0 - 0 = -3$

Thus required values of u_{ij} are tabulated

$t \backslash x$	0	1	2	3	4
0	0	3	4	3	0
0.5	0	2	3	2.	0
1	0	0	0	0	0
1.5	0	-2	-3	-2	0
2	0	-3	4	-3	0

(1m)

Q.No.	Solution and Scheme	Marks
8 C)	Solve the elliptic Eqn $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown. Find the first iterative values of u_i ($i = 1$ to 9) to the nearest integer  S.F – std 5 pt formula DF – Diagonal 5 pt formula	
Soln:	u_5 is located at the centre of the region and hence by the std 5 pt formula $u_5 = \frac{1}{4}[2000 + 2000 + 1000 + 1000]$ $u_5 = 1500$	(1m)
	Next we shall compute u_1, u_3, u_7, u_9 by the diagonal 5 pt formula $u_1 = \frac{1}{4}(0 + 1500 + 2000 + 1000) = 1125$ $u_3 = \frac{1}{4}(1000 + 2000 + 1500 + 0) = 1125$ $u_7 = 1125 = u_9$	(1m)
	Further we compute u_2, u_4, u_6, u_8 by S.F $u_2 = \frac{1}{4}[1125 + 1125 + 1000 + 1500] = 1187.5$ $u_4 = \frac{1}{4}[2000 + 1500 + 1125 + 1125] = 1437.5$ $u_6 = \frac{1}{4}[1500 + 2000 + 1125 + 1125] = 1437.5$ $u_8 = \frac{1}{4}[1125 + 1125 + 1500 + 1000] = 1187.5$	(1m)

These values are regarded as the initial approximations to commence the Liebmann's iteration we compute u_i $(i=1$ to $9)$ in the serial order by using the latest iteration values by applying S.F

$$u_1^{(1)} = \frac{1}{4}[1000 + 1187.5 + 500 + 1437.5] = 1031.25$$

$$u_2^{(1)} = \frac{1}{4}[1031.25 + 1125 + 1000 + 1500] = 1164.0625$$

$$u_3^{(1)} = \frac{1}{4}[1164.0625 + 1000 + 500 + 1437.5] = 1025.39$$

$$u_4^{(1)} = \frac{1}{4}[2000 + 1500 + 1031.25 + 1125] = 1414.0625$$

$$u_5^{(1)} = \frac{1}{4}[1414.0625 + 1437.5 + 1164.0625 + 1187.5] = 1300.7813$$ (3m)

$$u_6^{(1)} = \frac{1}{4}[1300.78 + 2000 + 1025.39 + 1125] = 1362.79$$

$$u_7^{(1)} = \frac{1}{4}[1000 + 1187.5 + 1414.0625 + 500] = 1025.39$$

$$u_8^{(1)} = \frac{1}{4}[1025.4 + 1125 + 1300.79 + 1000] = 1112.79$$

$$u_9^{(1)} = \frac{1}{4}[1112.79 + 1000 + 1362.79 + 500] = 993.89$$

Thus required first iterative values to the nearest integer are as follows

$u_1 = 1031$, $u_2 = 1164$, $u_3 = 1025$ $u_4 = 1414$

$u_5 = 1301$, $u_6 = 1363$ $u_7 = 1025$ (1m)

$u_8 = 1113$ $u_9 = 994$

Q.No.	Solution and Scheme	Marks
9(a)	R-K method	

By data $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, $y(0)=1$, $y'(0)=0$

Put $\frac{dy}{dx} = z \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx}$.

$\frac{dz}{dx} = x z^2 - y^2$, $y = 1$, $z = 0$ at $x = 0$.

Now we have system of Eqns

$f(x, y, z) = z$.

$g(x, y, z) = x z^2 - y^2$, $x_0 = 0$ $y_0 = 1$

$h = 0.2$, $z_0 = 0$ (1m)

We shall compute the following

$k_1 = h f(x_0, y_0\ z_0) = (0.2)\ f(0, 1, 0)$

$k_1 = (0.2)\ (0) = 0$

$l_1 = h\ g(x_0, y_0\ z_0) = 0.2\ [g(0, 1, 0)]$

$l_1 = 0.2\{0 - 1^2\} = -0.2$

$k_2 = h f\left(x_0 + \frac{h}{2}\ \ y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$k_2 = 0.2\ f(0.1, 1, -0.1) = (0.2)\ (-0.1)$ (2m)

$k_2 = -0.02$

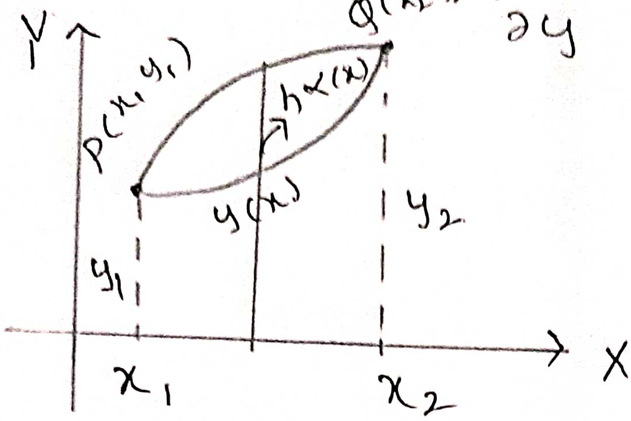
$l_2 = h\ g\{0.1, 1, -0.1\}$

$= 0.2\ [(0.1)\ (-0.1)^2 - 1^2] = -0.1998$

Q.No.	Solution and Scheme	Marks
	$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$	
	$k_3 = 0.2\ f(0.1, 0.99, -0.0999)$	
	$k_3 = (0.2)(-0.0999) = -0.01998$	
	$l_3 = h\ g(x_0, y_0, z_0) =$	(2m)
	$l_3 = (0.2)\ g(0.1, 0.99, -0.0999)$	
	$l_3 = 0.2[(0.1)(-0.0999)^2 - (0.99)^2] = -0.1958$	
	$k_4 = h f(x_0+h, y_0+k_3, z_0+l_3)$	
	$k_4 = 0.2\ f(0.2, 0.98002, -0.1958)$	
	$k_4 = (0.2)(-0.1958) = -0.03916$	
	$l_4 = h\ g(0.2, 0.98002, -0.1958)$	
	$l_4 = (0.2)[(0.2)(-0.1958)^2 - (0.98002)^2]$	
	$l_4 = -0.19055$	
	$k = \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$	
	$k = \frac{1}{6}\{0 + 2(-0.02) + 2(-0.01998) + (-0.03916)\}$	
	$k = -0.0199.$	
	$y(x_0+h) = y_0 + k = 1 - 0.0199$	
	$\boxed{y(0.2) = 0.9801}$	(1m)

Q.No. | **Solution and Scheme** | **Marks**

9b) Euler's Eqn $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$



(1m)

Let I be an Extremum along some curve $y = y(x)$ passing through $P(x_1, y_1)$ and $Q(x_2\ y_2)$

let $y = y(x) + h\alpha(x)$ —① be the neighbouring curve (where h is small) joining these points so that we must have $\alpha(x_1) = 0$ at P & $\alpha(x_2) = 0$ at Q ② when $h = 0$ these two curves coincide thus making I an Extremum ie

$$I = \int_{x_1}^{x_2} f\{x, y(x) + h\alpha(x), y'(x) + h\alpha'(x)\}dx$$

(2m)

is extremum when $h = 0$

This requires $\frac{dI}{dh} = 0$ when $h = 0$

treating I to be a function of h

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial}{\partial h} f\{x, y(x)+h\alpha(x), y'(x)+h\alpha'(x)\} dx.$$

Applying chain rule for partial derivative in RHS we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial h} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial h} \right\} dx \quad \text{③}$$

But h is independent of x $\Rightarrow \frac{\partial x}{\partial h} = 0$

Let us consider ① & diff w.r.t x

$$y' = y'(x) + h\alpha'(x) \quad \text{④}$$

we have from ① $\frac{\partial y}{\partial h} = \alpha(x)$ and

from (4) $\frac{\partial y'}{\partial h} = \alpha'(x)$

using these in Eqn ③

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \alpha(x) + \frac{\partial f}{\partial y'} \cdot \alpha'(x) \right] dx \quad \text{⑤}$$

(2m)

Integrating the 2nd term of RHS

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \alpha(x)\, dx \right] +$$

$$\left[\frac{\partial f}{\partial y'} \alpha(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \alpha(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) dx \Big]$$

Q.No.	Solution and Scheme	Marks
	$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \alpha(x)\,dx +$ $\left\{ \frac{\partial f}{\partial y} \alpha(x_2) - \frac{\partial f}{\partial y'} \alpha(x_1) \right\} - \int_{x_1}^{x_2} \alpha(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$	
	$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \alpha(x)\,dx$	
	$\because \alpha(x_1) = 0 = \alpha(x_2)$	
	But $\frac{dI}{dh} = 0$ when $h = 0$ for I to be Extremum. Hence integrand in the RHS must be zero since $\alpha(x)$ is arbitrary we must have	
	$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$	(2m)
	This is required Euler's Eqn.	
9c)	$f(x,y,y') = y^2 + y'^2 + 2ye^x$	(1m)
	Euler's Eqn $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$	(1m)
	$(2y + 2e^x) - \frac{d}{dx}(2y') = 0$	(1m)

Q.No.	Solution and Scheme	Marks
	$\Rightarrow y + e^{x} - y'' = 0$	(1m)
	$\Rightarrow y'' - y = e^{x}$ or $(D^2-1)\, y = e^{x}$ $\quad D = \frac{d}{dx}$	
	A E is $m^2 - 1 = 0 \Rightarrow m = \pm 1$	
	$y_c(x) = c_1 e^{x} + c_2 e^{-x}$	
	$y_p = \frac{e^{x}}{D^2 - 1}$ $\quad$ Put $D = 1$ in $f(D)$	
	$y_p = \frac{e^{x}}{1-1} = \frac{e^{x}}{0}$	
	$y_p = x \frac{e^{x}}{f'(D)} = x \cdot \frac{e^{x}}{2D} = \frac{x}{2} \frac{1}{D} e^{x}$	
	$y_p = \frac{x}{2} e^{x}$	
	$y = y_c + y_p$, $y = c_1 e^{x} + c_2 e^{-x} + \frac{x e^{x}}{2}$	(2m)

10 a) Milne's Predictor - Corrector method

Given Eqn $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$

$\frac{d^2y}{dx^2} = 2x + \frac{1}{2}\frac{dy}{dx}$ or $y'' = 2x + \frac{y'}{2}$

Put $y' = z \Rightarrow y'' = z'$

$\Rightarrow z' = 2x + \frac{z}{2}$ $\qquad h = 0.1$

x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
$y' = z$	$z_0 = 2$	$z_1 = 2.3178$	$z_2 = 2.6725$	$z_3 = 3.0657$
$y'' = z'$	$z_0' = 3$	$z_1' = 3.3589$	$z_2' = 3.73625$	$z_3' = 4.13285$

(1m)

consider Milne's Predictor formulae

$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$

$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$

$y_4^{(P)} = 2 + \frac{4}{3}(0.1)[2(2.3178) - 2.6725 + 2(3.0657)]$

$y_4^{(P)} = 3.0793$ (1m)

$z_4^{(P)} = 2 + \frac{4}{3}(0.1)[2(3.3589) - 3.73625 + 2(4.13285)]$

$z_4^{(P)} = 3.4996$ (1m)

consider Milne's corrector formulae

$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4^{(P)})$

$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$

$$y_4^{(c)} = 2\cdot4649 + \frac{1}{3}(0\cdot1)\left[2\cdot6725 + 4(3\cdot0657) + 2(3\cdot4996)\right]$$

$$y_4^{(c)} = \underline{3\cdot0794}$$ (1m)

$$z_4' = 2x_4 + \frac{z_4^{(p)}}{2} = 2(1\cdot4) + \frac{3\cdot4996}{2}$$

$$z_4' = 4\cdot5498$$

$$z_4^{(c)} = 2\cdot6725 + \frac{0\cdot1}{3}\left[3\cdot73625 + 4(4\cdot13285) + 4\cdot5498\right]$$

$$z_4^{(c)} = 3\cdot4997$$ (1m)

Applying corrector formula once again for $y_4^{(c)}$

$$y_4^{(c)} = 2\cdot4649 + \frac{0\cdot1}{3}\left[2\cdot6725 + 4(3\cdot0657) + 2(3\cdot4997)\right]$$

$$y_4^{(c)} = \underline{3\cdot0794}$$ (1m)

Thus $y(1\cdot4) = 3\cdot0794$

Q.No.	Solution and Scheme	Marks
10 b)	Prove that geodesics of a plane are straight lines	

Ans: let $y = y(x)$ be a curve joining two pts $P(x_1, y_1)$ & $Q(x_2, y_2)$ in the XOY plane. we know that the arc length betn P and Q is given by

$$S = \int_{x_1}^{x_2} \frac{ds}{dx}\, dx = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$ —— (1m)

ie $S = I = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$

we need to find the curve $y(x)$ such that I is minimum.

Let $f(x, y, y') = \sqrt{1+(y')^2}$ —— (1m)

Euler's Eqn $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 0 - \frac{d}{dx}\left\{\frac{1 \times \cancel{2}(y')}{\cancel{2}\sqrt{1+(y')^2}}\right\} = 0$$

$$\frac{d}{dx}\left\{\frac{y'}{\sqrt{1+(y')^2}}\right\} = 0$$ —— (1m)

$$\frac{\sqrt{1+(y')^2}\cdot y'' - y' \cdot \dfrac{1}{\cancel{2}\sqrt{1+(y')^2}}\,\cancel{2}y' \cdot y''}{1+(y')^2} = 0$$ —— (1m)

Q.No.	Solution and Scheme	Marks

$$\frac{[1+(y')^2]y'' - (y')^2 y''}{(1+[y']^2)\cdot\sqrt{1+(y')^2}} = 0$$

$$[1+(y')^2]y'' - (y')^2 y'' = 0$$

$$y'' + \cancel{(y')^2 y''} - \cancel{(y')^2 y''} = 0$$

$y'' = 0 \Rightarrow \frac{d^2y}{dx^2} = 0$ Integrating w.r.t x — (1 m)

$\frac{dy}{dx} = C_1$ Integrate w.r.t x once again

$y = C_1 x + C_2$ which is a straight line — (2m)

Hence geodesics on a plane are straight-lines.

Q.No.	Solution and Scheme	Marks
10 c)	Let $I = \int_0^1 [(y')^2 + 12xy]\, dx$	
	$f(x, y, y') = (y')^2 + 12xy$	(1m)
	Euler's Eqn $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$	(1m)
	$\frac{\partial f}{\partial y} = 12x$ $\frac{\partial f}{\partial y'} = 2y'$	
	$\Rightarrow 12x - \frac{d}{dx}(2y') = 0$ ie	
	$12x - 2y'' = 0 \Rightarrow \cancel{+}\cancel{2}y'' = \cancel{+}\overset{6}{\cancel{12}}x$	
	$y'' = 6x$ Integrating w.r.t x twice	(1m)
	$\frac{dy}{dx} = \overset{3}{\cancel{6}} \cdot \frac{x^2}{\cancel{2}} + c_1$	
	$y = \cancel{3} \cdot \frac{x^3}{\cancel{3}} + c_1 x + c_2$	
	$y(x) = x^3 + c_1 x + c_2$ —①	(1m)
	Using the given conditions	
	$y(0) = 0 \Rightarrow y = 0$ at $x = 0$	
	$y(1) = 1 \Rightarrow y = 1$ at $x = 1$	
	$y(0) = 0 + 0 + c_2 \Rightarrow \boxed{0 = c_2}$	(1m)
	$y(1) = 1 + c_1 + c_2 \Rightarrow 1 = 1 + c_1 + 0$	
	$\Rightarrow \boxed{c_1 = 0}$	(1m)
	substituting in Eqn ①	
	$y = x^3 + 0 + 0 \Rightarrow y = x^3$ is required soln	(1m)

Q.No.	Solution and Scheme	Marks
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