Model Question Paper-II with effect from 2022

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Third Semester B.E Degree Examination
Transform Calculus, Fourier Series and Numerical Techniques
(21MAT31)

TIME: 03 Hours **Max. Marks: 100**

Note: Answer any **FIVE** full questions, choosing at least **ONE** question from each module.

		Module -1	**Marks**
Q.01	a	Find the Laplace transform of $(i)\ e^{-3t} sin5t\ cos3t \qquad (ii)\ \frac{1-e^t}{t}$	**06**
	b	Find the Laplace transform of the square–wave function of period a given by $$f(t) = \begin{cases} 1, & 0 < t < a/2 \\ -1, & a/2 < t < 2 \end{cases}$$	**07**
	c	Using the convolution theorem find the inverse Laplace transform of $\frac{1}{(s^2+1)(s^2+9)}$	**07**
		OR	
Q.02	a	Using the unit step function, find the Laplace transform of $f(t) = \begin{cases} \cos t, & 0 \le t \le \pi \\ \cos 2t, & \pi \le t \le 2\pi \\ \cos 3t, & t \ge 2\pi \end{cases}$	**06**
	b	Find the inverse Laplace transform of $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$	**07**
	c	Solve by using Laplace transform techniques $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$ with $x(0) = 2$ and $x'(0) = -1$	**07**
		Module-2	
Q. 03	a	Find a Fourier series to represent $f(x) = x^2 \quad in \quad -\pi \le x \le \pi$	**06**
	b	Obtain the half-range cosine series for $f(x) = x\ sinx$ in $(0,\ \pi)$ and hence show that $\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty$	**07**
	c	The following table gives the variation of periodic current over a period. <table><tr><td>*t sec*</td><td>0</td><td>T/6</td><td>T/3</td><td>T/2</td><td>2T/3</td><td>5T/6</td><td>T</td></tr><tr><td>*A amp*</td><td>1.98</td><td>1.30</td><td>1.05</td><td>1.30</td><td>-0.88</td><td>-0.25</td><td>1.98</td></tr></table>Show that there is a direct current part of 0.75 amp in the variable current and obtain the amplitude of the first harmonic.	**07**

OR			
Q.04	a	Find the Fourier series expansion of $f(x) = 2x - x^2,\quad in\ (0,\ 3)$	**6**
	b	Obtain half-range sine series for $f(x) = \begin{cases} kx, & 0 \le x \le \frac{l}{2} \\ k(l-x), & \frac{l}{2} \le x \le l \end{cases}$	**07**
	c	Expand y as a Fourier series up to the first harmonic if the values of y are given by x: 0°, 30°, 60°, 90°, 120°, 150°, 180°, 210°, 240, 270, 300, 330 y: 1.80, 1.10, 0.30, 0.16, 1.50, 1.30, 2.16, 1.25, 1.30, 1.52, 1.76, 2.00	**07**
Module-3			
Q. 05	a	Find the Fourier transform of $f(x) = \begin{cases} 1, & \|x\| \le 1 \\ 0, & \|x\| > 1 \end{cases}$ Hence evaluate $\int_0^\infty \frac{sinx}{x} dx$	**06**
	b	Find the Fourier cosine and sine transforms of e^{-ax}	**07**
	c	Find the Z-transforms of $(i)\ (n+1)^2$ and $(ii)\ \sin(3n+5)$	**07**
OR			
Q. 06	a	Find the Fourier transform of $e^{-a^2x^2}, a > 0$. Hence deduce that it is self-reciprocal in respect of Fourier series	**06**
	b	Find the inverse z –transform of $\frac{2z^2+3z}{(z+2)(z-4)}$	**07**
	c	Using z-transformation, solve the difference equation $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$, $u_0 = 0, u_1 = 1$	**07**
Module-4			
Q. 07	a	Classify the following partial differential equations (i) $u_{xx} + 4u_{xy} + 4u_{yy} - u_x + 2u_y = 0$ (ii) $x^2u_{xx} + (1-y^2)u_{yy} = 0, -1 < y < 1$ (iii) $(1+x^2)u_{xx} + (5+2x^2)u_{xt} + (4+x^2)u_{tt} = 0$ (iv) $y^2u_{xx} - 2yu_{xy} + u_{yy} - u_y = 8y$	**10**
	b	Find the values of $u(x,t)$ satisfying the parabolic equation $u_t = 4u_{xx}$ and the boundary conditions $u(0,t) = 0 = u(8,0)$ and $u(x,0) = 4x - \frac{x^2}{2}$ at the points $x = i : i = 0,1,2,\dots,8$ and $t = \frac{j}{8} : j = 0,1,2,3,4.$	**10**
OR			
Q. 08	a	Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x,0) = \sin \pi x, 0 \le x \le 1$ $u(0,t) = u(1,t) = 0$, Carry out computations for two levels, taking $h = \frac{1}{34} and\ k = \frac{1}{36}$	**10**

	b	The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $u_{tt} = 25\, u_{xx},$ with the boundary conditions $u(x,t) = u(5,t) = 0$ and the initial conditions $u(x,0) = \begin{cases} 20x\,, & 0 \le x \le 1 \\ 5(5-x), & 1 \le x \le 5 \end{cases}$ and $u_t(x,0) = 0$. Solve this equation numerically up to $t = 5$ taking $h = 1, k = 0.2$.	**10**
		Module-5	
Q. 09	a	Using Runge –Kutta method of order four, solve $\frac{d^2y}{dx^2} = y + x\frac{dy}{dx}$ for x = 0.2 , Given that , $y(0) = 1\ \ , y'(0) = 0$	**06**
	b	Find the extremals of the functional $\int_{x_1}^{x_2}[y^2 + (y')^2 + 2ye^x]dx$	**07**
	c	Find the path on which a particle in the absence of friction, will slide from one point to another in the shortest time under the action of gravity	**07**
		OR	
Q. 10	a	Apply Milne's method to solve $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ at x = 0.4. given that $y(0) = 1\ , y(0.1) = 1.1103,\ y(0.2) = 1.2427\ ,\ y(0.3) = 1.399$ $y'(0) = 1\ \ , y'(0.1) = 1.2103, y'(0.2) = 1.4427, y'(0.3) = 1.699$	06
	b	Find the extremals of the functional $\int_{x_1}^{x_2}\frac{(y')^2}{x^3}dx$	**07**
	c	Find the curve on which the functional $\int_0^{\pi/2}[(y')^2 + 12xy]\,dx$ with $y(0) = 0\ and\ y(\pi/2) = 0$ can be extremised	**07**

Table showing the Bloom's Taxonomy Level, Course Outcome and Program Outcome

Question		Bloom's Taxonomy Level attached	Course Outcome	Program Outcome
Q.1	(a)	L1	CO 01	PO 01
	(b)	L2	CO 01	PO 02
	(c)	L2	CO 01	PO 02
Q.2	(a)	L2	CO 01	PO 02
	(b)	L2	CO 01	PO 02
	(c)	L2	CO 01	PO 02
Q.3	(a)	L2	CO 02	PO 02
	(b)	L2	CO 02	PO 02
	(c)	L3	CO 02	PO 02
Q.4	(a)	L2	CO 02	PO 02
	(b)	L2	CO 02	PO 02
	(c)	L2	CO 02	PO 02
Q.5	(a)	L2	CO 03	PO 02
	(b)	L2	CO 03	PO 02
	(c)	L2	CO 03	PO 02
Q.6	(a)	L2	CO 03	PO 02

	(b)	L2	CO 03	PO 02
	(c)	L3	CO 03	PO 02
Q.7	(a)	L1	CO 04	PO 01
	(b)	L2	CO 04	PO 02
Q.8	(a)	L2	CO 04	PO 02
	(b)	L3	CO 04	PO 02
Q.9	(a)	L2	CO 05	PO 01
	(b)	L2	CO 05	PO 02
	(c)	L3	CO 05	PO 02
Q.10	(a)	L2	CO 05	PO 01
	(b)	L2	CO 05	PO 02
	(c)	L2	CO 05	PO 02

Bloom's Taxonomy Levels	**Lower order thinking skills**		
	Remembering (knowledge): L_1	Understanding (Comprehension): L_2	Applying (Application): L_3
	Higher-order thinking skills		
	Analyzing (Analysis): L_4	Valuating (Evaluation): L_5	Creating (Synthesis): L_6

AY: 2022-23

Department: Mathematics **Model Question Paper - II**
Branch: EC/CV/EE/CS/ME **Semester: III**
Subject with Sub. Code: Transform Calculus, Fourier Series and Numerical Techniques (21MAT31)
Name of Faculty: Dr. Lata Lamani

Q.No.	Solution and Scheme	Marks
1. (a)	(i) To find: $L[e^{-3t}\sin 5t\cos 3t]$	
	WKT, $\sin 5t\cos 3t = \frac{\sin(5t+3t)+\sin(5t-3t)}{2}$	
	$\left[\because \sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}\right]$	
	$\therefore \sin 5t\cos 3t = \frac{\sin 8t+\sin 2t}{2}$	(2M)
	Taking Laplace transforms on both sides, we have	
	$L[\sin 5t\cos 3t] = \frac{1}{2}\left\{L[\sin 8t]+L[\sin 2t]\right\}$	(1M)
	ie $L[\sin 5t\cos 3t] = \frac{1}{2}\left\{\frac{8}{s^2+64}+\frac{2}{s^2+4}\right\}$	(1M)
	WKT, $L[e^{at}f(t)] = \bar{f}(s-a)$ or $L[e^{-at}f(t)] = \bar{f}(s+a)$	(1M)
	$\therefore L[e^{-3t}\sin 5t\cos 3t] = \frac{1}{2}\left\{\frac{8}{(s+3)^2+64}+\frac{2}{(s+3)^2+4}\right\}$	(1M)
		(6M)
(b)	Given: $f(t) = \begin{cases} 1, & 0<t<a/2 \\ -1, & a/2<t<a \end{cases}$	
	The given function is periodic with period $T=a$.	
	We have, $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$	(1M)
	$= \frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)\,dt$	
	$= \frac{1}{1-e^{-as}}\int_0^{a/2} e^{-st}\,dt + \frac{1}{1-e^{-as}}\int_{a/2}^{a} -e^{-st}\,dt$	(1M)

Q.No.	Solution and Scheme	Marks

$$\therefore L[f(t)] = \frac{1}{1-e^{-as}}\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \frac{1}{1-e^{-as}}\left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}$$

$$= \frac{1}{1-e^{-as}}\left[\frac{e^{-as/2}-1}{-s} - \frac{e^{-as}-e^{-as/2}}{-s}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{1-e^{-as/2}-e^{-as/2}+e^{-as}}{s}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{1-2e^{-as/2}+e^{-as}}{s}\right]$$ (1M)

$$= \frac{1}{1-e^{-as}}\left[\frac{1-2e^{-as/2}+(e^{-as/2})^2}{s}\right]$$

$$= \frac{1}{1-(e^{-as/2})^2}\left[\frac{(1-e^{-as/2})^2}{s}\right]$$

$$= \frac{1}{\cancel{(1-e^{-as/2})}(1+e^{-as/2})}\left[\frac{(1-e^{-as/2})^{\cancel{2}}}{s}\right]$$ (1M)

(1M)

$$\therefore L[f(t)] = \frac{1}{s}\left[\frac{1-e^{-as/2}}{1+e^{-as/2}}\right]$$

Multiplying and Dividing RHS by $e^{as/4}$, we get

$$L[f(t)] = \frac{1}{s}\left[\frac{e^{as/4}-e^{-as/4}}{e^{as/4}+e^{as/4}}\right]$$ (1M)

$$\therefore L[f(t)] = \frac{1}{s}\tanh\left(\frac{as}{4}\right) \qquad \left[\because \tanh x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]$$ (1M)

(7M)

(c) Let $\bar{f}(s) = \frac{1}{(s^2+1)(s^2+9)}$

Now, $\frac{1}{(s^2+1)(s^2+9)} = \frac{1}{s^2+1}\cdot\frac{1}{s^2+9}$

ie $\frac{1}{(s^2+1)(s^2+9)} = \bar{f}(s)\cdot\bar{g}(s)$ (1M)

Q.No.	Solution and Scheme	Marks
	Taking inverse Laplace transforms on both sides,	
	$$L^{-1}\left[\frac{1}{(s^2+1)(s^2+9)}\right] = L^{-1}\left[\bar{f}(s)\cdot\bar{g}(s)\right] \quad \text{——} (1)$$	
	Now, $\bar{f}(s) = \frac{1}{s^2+1}$ and $\bar{g}(s) = \frac{1}{s^2+9}$	
	ie $f(t) = \sin t$ and $g(t) = \frac{\sin 3t}{3}$	
	ie $f(u) = \sin u$ and $g(t-u) = \frac{\sin(3t-3u)}{3}$	
	By Convolution theorem, we have	
	$$\int_{u=0}^{t} f(t)\cdot g(t-u)\,du = L^{-1}\left[\bar{f}(s)\cdot\bar{g}(s)\right]$$	(2M)
	ie $L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+9)}\right] = \int_{u=0}^{t} \sin u \cdot \frac{\sin(3t-3u)}{3}\,du$	(1M)
	$$= \frac{1}{3}\int_{u=0}^{t} \sin u \cdot \sin(3t-3u)\cdot du$$	
	$$= \frac{1}{3}\int_{u=0}^{t}\left[\frac{\cos(u-3t+3u)-\cos(u+3t-3u)}{2}\right]$$	
	$$= \frac{1}{3}\int_{u=0}^{t}\left[\frac{\cos(4u-3t)-\cos(3t-2u)}{2}\right]du$$	
	$$= \frac{1}{6}\left[\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{-2}\right]_{u=0}^{t}$$	
	$$= \frac{1}{24}\left[\sin(4u-3t)+2\sin(3t-2u)\right]_{u=0}^{t}$$	

Q.No.	Solution and Scheme	Marks
	$\therefore L^{-1}\left[\dfrac{1}{(s^2+1)(s^2+9)}\right] = \dfrac{1}{24}\left[\sin t + 2\sin t + \sin 3t - 2\sin 3t\right]$	(2M)
	ie $L^{-1}\left[\dfrac{1}{(s^2+1)(s^2+9)}\right] = \dfrac{1}{24}\left[3\sin t - \sin 3t\right]$	(1M)
		(7M)
Q.02		
(a)	Given: $f(t) = \begin{cases} \cos t, & 0 \le t \le \pi \\ \cos 2t, & \pi \le t \le 2\pi \\ \cos 3t, & t \geq 2\pi \end{cases}$	
	$\therefore f(t) = \cos t + (\cos 2t - \cos t)u(t-\pi) + (\cos 3t - \cos 2t)u(t-2\pi)$	(1M)
	ie $L[f(t)] = L[\cos t] + L[(\cos 2t - \cos t)u(t-\pi)] + L[(\cos 3t - \cos 2t)u(t-2\pi)]$ — (1)	
	Let $F(t-\pi) = \cos 2t - \cos t$; $G(t-2\pi) = \cos 3t - \cos 2t$	
	$\therefore F(t) = \cos 2(t+\pi) - \cos(t+\pi)$; $G(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$	
	ie $F(t) = \cos 2t + \cos t$; $G(t) = \cos 3t - \cos 2t$	
	ie $\bar{F}(s) = \dfrac{s}{s^2+4} + \dfrac{s}{s^2+1}$; $\bar{G}(s) = \dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}$	(1M)
	WKT, $L[F(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{F}(s)$ and $L[G(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\bar{G}(s)$	(1M)
	ie $L[(\cos 2t - \cos t)u(t-\pi)] = e^{-\pi s}\left[\dfrac{s}{s^2+4} + \dfrac{s}{s^2+1}\right]$ and $L[(\cos 3t - \cos 2t)u(t-2\pi)] = e^{-2\pi s}\left[\dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}\right]$	(1M)
	$\therefore$ From (1),	
	$L[f(t)] = \dfrac{s}{s^2+1} + e^{-\pi s}\left[\dfrac{s}{s^2+4} + \dfrac{s}{s^2+1}\right] + e^{-2\pi s}\left[\dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}\right]$	(1M)

Q.No.	Solution and Scheme	Marks
	ie $L[f(t)] = \frac{s}{s^2+1} + se^{-\pi s}\left[\frac{2s^2+5}{(s^2+1)(s^2+4)}\right] + e^{-2\pi s}\left[\frac{5s}{(s^2+4)(s^2+9)}\right]$	(1M) (6M)
(b)	To find: $L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]$	
	$\rightarrow \frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s^2-5s+6)} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$	(2M)
	Now, $\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$ ie $2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$	
	Put $s=1$, $1 = A(-1)(-2)$, $\boxed{A = \frac{1}{2}}$ Put $s=2$, $1 = B(1)(-1)$, $\boxed{B = -1}$ Put $s=3$, $5 = C(2)(1)$, $\boxed{C = \frac{5}{2}}$	(2M)
	ie $\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{1}{2(s-1)} - \frac{1}{s-2} + \frac{5}{2(s-3)}$	(2M)
	ie $L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] = \frac{e^t}{2} - e^{2t} + \frac{5e^{3t}}{2}$	(1M) (7M)
(c)	Given: $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$ with $x(0)=2$ and $x'(0)=-1$. ie $x''(t) - 2x'(t) + x(t) = e^t$ Taking Laplace transform on both sides, $L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^t]$	(1M)

Q.No.	Solution and Scheme	Marks
	ie $s^2L[x(t)]-sx(0)-x'(0)-2\{sL[x(t)]-x(0)]+L[x(t)]=\frac{1}{s-1}$	
	ie $s^2L[x(t)]-2s+1-2sL[x(t)]+4+L[x(t)]=\frac{1}{s-1}$	
	ie $L[x(t)]\{s^2-2s+1\}=\frac{1}{s-1}+2s-5$	
	ie $L[x(t)](s-1)^2=\frac{1}{s-1}+2s-5$	(1M)
	ie $L[x(t)]=\frac{1}{(s-1)^3}+\frac{2s}{(s-1)^2}-\frac{5}{(s-1)^2}$	(1M)
	ie $L[x(t)]=\frac{1}{(s-1)^3}+2\left[\frac{(s-1)+1}{(s-1)^2}\right]-\frac{5}{(s-1)^2}$	
	ie $L[x(t)]=\frac{1}{(s-1)^3}+2\left[\frac{(s-1)}{(s-1)^2}+\frac{1}{(s-1)^2}\right]-\frac{5}{(s-1)^2}$	
	ie $L[x(t)]=\frac{1}{(s-1)^3}+2\left[\frac{1}{(s-1)}+\frac{1}{(s-1)^2}\right]-\frac{5}{(s-1)^2}$	
	$\therefore x(t)=L^{-1}\left[\frac{1}{(s-1)^3}\right]+2L^{-1}\left[\frac{1}{s-1}\right]+2L^{-1}\left[\frac{1}{(s-1)^2}\right]-5L^{-1}\left[\frac{1}{(s-1)^2}\right]$	(1M)
	$=L^{-1}\left[\frac{1}{(s-1)^3}\right]+2L^{-1}\left[\frac{1}{s-1}\right]-3L^{-1}\left[\frac{1}{(s-1)^2}\right]$	
	$=e^tL^{-1}\left[\frac{1}{s^3}\right]+2e^tL^{-1}\left[\frac{1}{s}\right]-3e^tL^{-1}\left[\frac{1}{s^2}\right]$	
	$=\frac{e^tt^2}{2}+2e^t-3te^t$	
	$=e^t\left[\frac{t^2}{2}-3t+2\right]$	(2M)
		(1M)
	$\therefore x(t)=\underline{\underline{\frac{e^t}{2}\left[t^2-6t+4\right]}}$	(7M)

Q.No.	Solution and Scheme	Marks
Q.03 (a)	The Fourier series of $f(x)$ in $-\pi \le x \le \pi$ is given by:	
	$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (1)}$$	
	Given: $f(x) = x^2$	
	$\therefore f(-x) = (-x)^2 = x^2 = f(x)$	
	ie $f(-x) = f(x)$	(1M)
	Hence $f(x)$ is even and therefore $b_n = 0$.	
	$\therefore$ Equation (1) becomes:	(1M)
	$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—— (2)}$$	
	Where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$	
	$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$	
	$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$	
	$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$	
	$\therefore a_0 = \frac{2\pi^2}{3}$	
	ie $\frac{a_0}{2} = \frac{\pi^2}{3}$ —— (3)	(1M)
	Now, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$	
	$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$	
	$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$	
	$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$	

Q.No.	Solution and Scheme	Marks
	$\therefore a_n = \dfrac{4(-1)^n}{n^2}$	
	ie $a_n = 4\left[\dfrac{(-1)^n}{n^2}\right]$ —— (3)	(2M)
	Substituting (3) and (4) in (1), we get	
	$f(x) = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx$	(1M)
	is the required Fourier series of $f(x)=x^2$ in $-\pi \le x \le \pi$	(6M)
(b)	The half range Cosine series for $f(x)$ in $(0,\pi)$ is given by:	
	$f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ —— (1)	(1M)
	Where, $a_0 = \dfrac{2}{\pi}\int_0^{\pi} f(x)\,dx$	
	$= \dfrac{2}{\pi}\int_0^{\pi} x\sin x\,dx$	
	$= \dfrac{2}{\pi}\Big[x(-\cos x) - (1)(-\sin x)\Big]_0^{\pi}$	
	$= \dfrac{2}{\pi}\Big[-\pi(-1)\Big]$	
	$a_0 = 2(1)$	
	ie $\dfrac{a_0}{2} = 1$ —— (2)	(1M)
	Now, $a_n = \dfrac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$	
	$= \dfrac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$	
	$= \dfrac{2}{\pi}\int_0^{\pi} x\left\{\dfrac{\sin(1+n)x + \sin(1-n)}{2}\right\}dx$	
	$= \dfrac{1}{\pi}\int_0^{\pi} x\left\{\sin(1+n)x + \sin(1-n)x\right\}dx$	

Q.No.	Solution and Scheme	Marks
	$\therefore a_n = \frac{1}{\pi}\int_0^{\pi} x\left\{\sin(n+1)x - \sin(n-1)x\right\}dx.$	
	$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$	
	$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$	
	$= \frac{1}{\pi}\left[\frac{(-1)(-1)^n(-1)}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{n-1-n-1}{n^2-1}\right]$	
	$= \frac{1}{\pi}\left[\frac{(n-1)(-1)^n + (-1)^n(-1)^{-1}(n+1) - 2}{n^2-1}\right]$	
	$= \frac{1}{\pi}\left[\frac{n(-1)^n - (-1)^n - (-1)^n n - (-1)^n - 2}{n^2-1}\right]$	
	$= \frac{1}{\pi}\left[\frac{-2(-1)^n - 2}{n^2-1}\right]$	
	$= \frac{1}{\pi}\left[\frac{2(-1)^{n+1} - 1}{n^2-1}\right]$	
	$\therefore a_n = \frac{2}{\pi}\left[\frac{(-1)^{n+1} - 1}{n^2-1}\right], \; n \neq 1$ —— (3)	(2M)
	For $n=1$, $a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx$	
	$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$	
	$= \frac{2}{\pi}\int_0^{\pi} x\left\{\frac{\sin 2x}{2}\right\}dx$	
	$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$	
	$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$	
	$= \frac{1}{\pi}\left[\pi\left(\frac{-1}{2}\right)\right]$	
	$\therefore a_1 = \frac{-1}{2}$ —— (4)	(2M)

Q.No.	Solution and Scheme	Marks
	Substituting (2), (3) and (4) in (1), we get	
	$f(x) = 1 - \frac{\cos x}{2} + \frac{2}{\pi} \sum_{n=}^{\infty} \left[\frac{(-1)^{n+1} - 1}{n^2 - 1} \right] \cos nx$	(1M)
		(7M)
(c)	Given: The following table gives the variation of periodic Current over a period.	
	t sec: 0, T/6, T/3, T/2, 2T/3, 5T/6, T A amp: 1.98, 1.30, 1.05, 1.30, −0.88, −0.25, 1.98	
	We observe that the values of A at t=0 and t=T are the same. Hence we shall omit the last value. We Convert A=f(t) to the period 2π by putting $\theta = \left(\frac{2\pi t}{T}\right)$ so that we have $\theta = 0$ when $t=0$. The Corresponding values of 'θ' are respectively $0°, 60°, 120°, 180°, 240°, 300°, 360°$.	(1M)
	The Fourier series upto the first harmonics is represented by: $A = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta)$	(1M)
	We prepare the relevant table Considering the values of 'A' and 'θ' in $0 \le \theta \le 2\pi$.	

t	A	$\theta°$	$\cos\theta$	$A\cos\theta$	$\sin\theta$	$A\sin\theta$
0	1.98	0	1	1.98	0	0
T/6	1.30	60	0.5	0.65	0.866	1.1258
T/3	1.05	120	−0.5	−0.525	0.866	0.9093
T/2	1.30	180	−1	−1.30	0	0
2T/3	−0.88	240	−0.5	0.44	−0.866	0.76208
5T/6	−0.25	300	0.5	−0.125	−0.866	0.2165
Total	4.5			1.12		3.01368

(2M)

Q.No.	Solution and Scheme	Marks
	Now, $a_0=\frac{2}{N}\sum A=\frac{1}{3}(4.5)=1.5$; $\frac{a_0}{2}=0.75$	
	$a_1=\frac{2}{N}\sum A\cos x=\frac{1}{3}(1.12)=0.3733$	(1M)
	$b_1=\frac{2}{N}\sum A\sin x=\frac{1}{3}(3.01368)=1.00456$	
	The required Fourier series upto the first harmonics is given by:	
	$A=0.75+(0.3733\cos x+1.00456\sin x)$	(1M)
	The direct current part of the variable current is the Constant term in the Fourier series being 0.75.	(1M)
	Amplitude of first harmonic $=\sqrt{a_1^2+b_1^2}=1.072$.	(1M)
		(7M)
Q.04		
(a)	Here $(0,2l)=(0,3)$ ie $2l=3$ or $l=3/2$	(1M)
	The Fourier series of $f(x)$ having period $(0,2l)$ is given by:	
	$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right)$	
	ie $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{2n\pi x}{3}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{2n\pi x}{3}\right)$ $[\because l=3/2]$ ——(1)	(1M)
	Where, $a_0=\frac{1}{l}\int_0^{2l}f(x)\,dx$	
	$=\frac{2}{3}\int_0^{3}(2x-x^2)\,dx$ $[\because l=3/2]$	

Q.No.	Solution and Scheme	Marks
	$\therefore a_0 = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$	
	$= \frac{2}{3}\left[3^2 - \frac{27}{3}\right]$	
	$= \frac{2}{3}\left[9-9\right]$	
	$\therefore a_0 = 0$	
	ie $\frac{a_0}{2} = 0$ ——— (2)	(1M)
	Now, $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$	
	$= \frac{2}{3}\int_0^3 (2x-x^2)\cos\left(\frac{2n\pi x}{3}\right)dx \quad \left[\because l = 3/2\right]$	
	$= \frac{2}{3}\left[(2x-x^2)\left\{\sin\left(\frac{2n\pi x}{3}\right)\right\}\left(\frac{3}{2n\pi}\right) - (2-2x)\left\{-\cos\left(\frac{2n\pi x}{3}\right)\right\}\left(\frac{9}{4n^2\pi^2}\right) + (-2)\left\{-\sin\left(\frac{2n\pi x}{3}\right)\right\}\left(\frac{27}{8n^3\pi^3}\right)\right]_0^3$	
	$= \frac{2}{3}\left[(2-6)\left\{\cos 2n\pi\right\}\left(\frac{9}{4n^2\pi^2}\right) - (2-0)(1)\left(\frac{9}{4n^2\pi^2}\right)\right]$	
	$= \frac{2}{3}\left[\frac{9}{4n^2\pi^2}\left\{-4-2\right\}\right]$	
	$= \frac{-2}{3}\left[\frac{9\cdot 6}{4n^2\pi^2}\right]$	
	$\therefore a_n = \frac{-9}{n^2\pi^2}$ ——— (3)	(1M)

Q.No.	Solution and Scheme	Marks
	Now, $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$	
	ie $b_n = \frac{2}{3}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)dx \quad \left[\because l=\frac{3}{2}\right]$	
	$= \frac{2}{3}\Big[(2x-x^2)\left\{-\cos\left(\frac{2n\pi x}{3}\right)\right\}\left(\frac{3}{2n\pi}\right)$ $-(2-2x)\left\{-\sin\left(\frac{2n\pi x}{3}\right)\right\}\left(\frac{9}{4n^2\pi^2}\right)$ $+(-2)\left\{\cos\left(\frac{2n\pi x}{3}\right)\right\}\left(\frac{27}{8n^3\pi^3}\right)\Big]_0^3$	
	$= \frac{2}{3}\left[(6-9)(-1)\left(\frac{3}{2n\pi}\right) - 2(1)\left(\frac{27}{8n^3\pi^3}\right) + 2\left(\frac{27}{8n^3\pi^3}\right)\right]$	
	$= \frac{2}{3}\left[3\left(\frac{3}{2n\pi}\right)\right]$	
	$\therefore b_n = \frac{3}{n\pi}$ ——— (4)	(1M)
	Substituting (2), (3) and (4) in (1), we get	
	$f(x) = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$	(1M)
	is the required Fourier series of the given function $f(x)=2x-x^2$ in $(0,3)$.	(6M)
(b)	The half-range sine series of $f(x)$ in $(0,l)$ is given by:	
	$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ ——— (1)	(1M)

Q.No.	Solution and Scheme	Marks
	Where, $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$	(1M)
	$= \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$	
	$= \frac{2}{l}\left[\int_0^{l/2} kx\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} k(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$	
	$= \frac{2k}{l}\left[\left\{x\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right) - (1)\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\cdot\frac{l^2}{n^2\pi^2}\right\}_0^{l/2} + \left\{(l-x)\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right) - (-1)\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\cdot\frac{l^2}{n^2\pi^2}\right\}_{l/2}^{l}\right]$	
	$= \frac{2k}{l}\left[\frac{l}{2}\cdot\frac{l}{n\pi}\left\{-\cos\left(\frac{n\pi}{2}\right)\right\} + \frac{l^2}{n^2\pi^2}\left\{\sin\left(\frac{n\pi}{2}\right)\right\} + \frac{l}{2}\cdot\frac{l}{n\pi}\left\{\cos\left(\frac{n\pi}{2}\right)\right\} + \frac{l^2}{n^2\pi^2}\left\{\sin\left(\frac{n\pi}{2}\right)\right\}\right]$	
	$= \frac{2k}{l}\left[\frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$	(2M)
	$\therefore b_n = \frac{4kl}{\pi^2}\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$ —— (2)	(2M)
	Substituting (2) in (1), we get	
	$f(x) = \frac{4kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin nx$	(1M)
	is the sine half range series for $f(x)$ in $(0, l)$.	(7M)

Q.No.	Solution and Scheme	Marks
(c)	Given:	

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

Here the interval of 'x' is $0 \le x \le 2\pi$. Period of y is 2π

We are required to find a_0, a_1, b_1. The Corresponding formulae are:

$a_0 = \frac{2}{N}\sum y$, $a_1 = \frac{2}{N}\sum y\cos x$, $b_1 = \frac{2}{N}\sum y\sin x$, $N=12$, $\frac{2}{N} = \frac{1}{6}$ (1M)

x	y	$\cos x$	$y\cos x$	$\sin x$	$y\sin x$
0	1.80	1	1.80	0	0
30°	1.10	0.866	0.9526	0.5	0.55
60°	0.30	0.5	0.15	0.866	0.2598
90°	0.16	0	0	1	0.16
120°	1.50	−0.5	−0.75	0.866	1.299
150°	1.30	−0.866	−1.1258	0.5	0.65
180°	2.16	−1	−2.16	0	0
210°	1.25	−0.866	−1.0825	−0.5	−0.625
240°	1.30	−0.5	−0.65	−0.866	−1.1258
270°	1.52	0	0	−1	−1.52
300°	1.76	0.5	0.88	−0.866	−1.52416
330°	2.00	0.866	1.732	−0.5	−1
Total	16.15		−0.2537		−2.87616

(2M)

Now, $a_0 = \frac{1}{6}\sum y = \frac{1}{6}(16.15) = 2.6917$; $\frac{a_0}{2} = 1.34585$ (1M)

$a_1 = \frac{1}{6}\sum y\cos x = \frac{1}{6}(-0.2537) = -0.0423$ (1M)

$b_1 = \frac{1}{6}\sum y\sin x = \frac{1}{6}(-2.87616) = -0.47936$ (1M)

The Fourier series upto the first harmonics is given by

$$y = f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x)$$

Thus, $y = 1.34585 - 0.0423\cos x - 0.47936\sin x$ (1M)

(7M)

Q.No.	Solution and Scheme	Marks
Q.05 (a)	Complex Fourier transform of $f(x)$ is given by: $F(u)=\int_{x=-\infty}^{\infty} f(x)e^{iux}\,dx$	(1M)
	ie $F(u)=\int_{x=-1}^{1} 1.\,e^{iux}\,dx$, since $f(x)=\begin{cases}1 & \text{for } -1\le x\le 1\\ 0 & \text{otherwise}\end{cases}$	
	$F(u)=\left[\dfrac{e^{iux}}{iu}\right]_{x=-1}^{1}$	
	$=\dfrac{1}{iu}\left[e^{iu}-e^{-iu}\right]$	
	$=\dfrac{1}{iu}\left[\{\cos u+i\sin u\}-\{\cos u-i\sin u\}\right]$	
	$=\dfrac{1}{iu}(2i\sin u)$	
	$\therefore F(u)=\dfrac{2\sin u}{u}$	(1M)
	Next, to evaluate: $\int_0^{\infty}\dfrac{\sin x}{x}\,dx$	
	We have, $F(u)=\dfrac{2\sin u}{u}$	
	Inverse Fourier transform is: $\dfrac{1}{2\pi}\int_{-\infty}^{\infty}F(u)\,e^{-iux}\,dx=f(x)$	
	ie $f(x)=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}\dfrac{2\sin u}{u}\,e^{-iux}\,dx=\dfrac{1}{\pi}\int_{-\infty}^{\infty}\dfrac{\sin u}{u}\,e^{-iux}\,du$	(1M)
	Now, let us put $x=0$. Since $x=0$ is the point of Continuity of $f(x)$, the value of $f(x)$ at $x=0$ being $f(0)=1$ since $f(x)=1$ for $\lvert x\rvert\le 1$.	(1M)

Q.No.	Solution and Scheme	Marks
	Hence, $\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin u}{u}du=1$, since $e^{0}=1$.	
	ie $\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin u}{u}du=1$, since $\frac{\sin u}{u}$ is an even function of 'u'.	
	$\therefore \int_{0}^{\infty}\frac{\sin u}{u}du=\frac{\pi}{2}$	(1M)
	Changing the variable from 'u' to 'x', we get	
	$\int_{0}^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$	(1M)
		(6M)
(b)	Fourier Cosine and sine transforms are given by:	
	$F_c(u)=\int_{0}^{\infty}f(x)\cos ux\,dx$ and $F_s(u)=\int_{0}^{\infty}f(x)\sin ux\,dx$.	(1M)
	$\therefore F_c(u)=\int_{0}^{\infty}e^{-ax}\cos ux\,dx$	(1M)
	$=\left[\frac{e^{-ax}}{(-a)^2+u^2}(-a\cos ux+u\sin ux)\right]_{x=0}^{\infty}$	(1M)
	$\therefore F_c(u)=\frac{a}{a^2+u^2}$	(1M)
	Now, $F_s(u)=\int_{0}^{\infty}e^{-ax}\sin ux\,dx$	(1M)
	$=\left[\frac{e^{-ax}}{(-a)^2+u^2}(-a\sin ux-u\cos ux)\right]$	(1M)
	$\therefore F_s(u)=\frac{u}{a^2+u^2}$	(1M)
		(7M)

Q.No.	Solution and Scheme	Marks
(c)	(i) $Z_T\left[(n+1)^2\right] = Z_T\left[n^2+2n+1\right]$	(1M)
	ie $Z_T\left[(n+1)^2\right] = Z_T\left(n^2\right) + 2Z_T(n) + Z_T(1)$	
	WKT, $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$, $Z_T(n) = \dfrac{z}{(z-1)^2}$, $Z_T(1) = \dfrac{z}{z-1}$	
	$\therefore Z_T\left[(n+1)^2\right] = \dfrac{z^2+z}{(z-1)^3} + \dfrac{2z}{(z-1)^2} + \dfrac{z}{z-1}$	
	$= \dfrac{z^2+z+2z(z-1)+z(z-1)^2}{(z-1)^3}$	
	$= \dfrac{z^2+\cancel{z}+\cancel{2z^2}-2z+z^3-\cancel{2z^2}+\cancel{z}}{(z-1)^3}$	(1M)
	$= \dfrac{z^3+z^2}{(z-1)^3}$	
	$\therefore Z_T\left[(n+1)^3\right] = \dfrac{z^2(z+1)}{(z-1)^3}$	(1M)
	(ii) Let $u_n = \sin(3n+5) = \sin 3n \cos 5 + \cos 3n \sin 5$	
	$\therefore Z_T(u_n) = \cos 5 \cdot Z_T[\sin 3n] + \sin 5 \cdot Z_T[\cos 3n]$ —— (1)	(1M)
	Consider, $e^{i(3n)} = (e^{3i})^n = k^n$ (say) Where $k = e^{3i}$	
	WKT, $Z[k^n] = \dfrac{z}{z-k}$	
	ie $Z(e^{3in}) = \dfrac{z}{z-e^{3i}} = \dfrac{z}{(z-\cos 3)-i\sin 3}$	
	$= \dfrac{z\left[(z-\cos 3)+i\sin 3\right]}{(z-\cos 3)^2+\sin^2 3}$	

Q.No.	Solution and Scheme	Marks
	ie $Z_T(\cos 3n + i\sin 3n) = \dfrac{z(z-\cos 3)}{z^2-2z\cos 3+1} + i\dfrac{z\sin 3}{z^2-2z\cos 3+1}$	
	$\Rightarrow Z_T(\cos 3n) = \dfrac{z(z-\cos 3)}{z^2-2z\cos 3+1}$ and $Z_T(\sin 3n) = \dfrac{z\sin 3}{z^2-2z\cos 3+1}$	
	Substituting these results in (1), we get	
	$Z_T(u_n) = \cos 5 . \dfrac{z\sin 3}{z^2-2z\cos 3+1} + \sin 5 . \dfrac{z(z-\cos 3)}{z^2-2z\cos 3+1}$	
	$= \dfrac{z(\sin 3\cos 5 - \cos 3\sin 5) + z^2\sin 5}{z^2-2z\cos 3+1}$	(2M)
	$= \dfrac{z(-\sin 2)+z^2\sin 5}{z^2-2z\cos 3+1}$	
	$\therefore Z_T(u_n) = \dfrac{z(\sin 5 - \sin 2)}{z^2-2z\cos 3+1}$	
	ie $Z_T(\sin(3n+5)) = \dfrac{z(\sin 5-\sin 2)}{z^2-2z\cos 3+1}$	(1M)
		(7M)
Q.06		
(a)	$F(u) = F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{iux}\,dx$	(1M)
	ie $F(u) = \int_{-\infty}^{\infty} e^{-a^2x^2} . e^{iux}\,dx$	
	$= \int_{-\infty}^{\infty} e^{-a^2\left(x^2 - \frac{iux}{a^2}\right)}\,dx$	
	$= \int_{-\infty}^{\infty} e^{-a^2\left(x^2 - 2x.\frac{iu}{2a^2} + \frac{i^2u^2}{4a^4} - \frac{i^2u^2}{4a^4}\right)}\,dx$	
	$= \int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{iu}{2a^2}\right)^2} . e^{-u^2/4a^2}\,dx$	(1M)

Q.No.	Solution and Scheme	Marks
	Put $a\left(x-\frac{iu}{2a^2}\right)=t \therefore dx=\frac{dt}{a}$ and 't' also varies from $-\infty$ to ∞.	(1M)
	Now, $F(u)=e^{-u^2/4a^2}\int_{-\infty}^{\infty} e^{-t^2}\cdot\frac{dt}{a}$ and We know that $\int_{-\infty}^{\infty} e^{-t^2}dt=\sqrt{\pi}$	
	Thus, $F(u)=\frac{\sqrt{\pi}}{a}e^{-u^2/4a^2}$	(1M)
	Taking $a^2=1/2$, we have	
	$F(u)=F\left[e^{-x^2/2}\right]=\frac{\sqrt{\pi}}{(1/\sqrt{2})}e^{-u^2/2}=2\sqrt{\pi}\,e^{-u^2/2}$	(1M)
	It can be seen that the Fourier transform of $e^{-x^2/2}$ is a Constant times $e^{-u^2/2}$	(1M)
	The function $e^{-x^2/2}$ and $e^{-u^2/2}$ are same but for the change in variable.	
	Hence we Conclude that $e^{-x^2/2}$ is a self reciprocal under Complex Fourier transform.	(6M)
(b)	Let $\overline{u}(z)=\frac{2z^2+3z}{(z+2)(z-4)}$	(1M)
	$\therefore \frac{\overline{u}(z)}{z}=\frac{2z+3}{(z+2)(z-4)}$	(1M)
	Let $\frac{2z+3}{(z+2)(z-4)}=\frac{A}{z+2}+\frac{B}{z-4}$	
	ie $2z+3=A(z-4)+B(z+2)$	
	Put $z=-2 \Rightarrow -1=A(-6)\Rightarrow \boxed{A=1/6}$	(1M)
	$z=4\Rightarrow 11=B(6)\Rightarrow \boxed{B=\frac{11}{6}}$	(1M)

Q.No.	Solution and Scheme	Marks

Hence, $\dfrac{\bar{u}(z)}{z} = \dfrac{1}{6}\cdot\dfrac{1}{z+2} + \dfrac{11}{6}\cdot\dfrac{1}{z-6}$

$\underline{\text{Or}}$ $\bar{u}(z) = \dfrac{1}{6}\left[\dfrac{z}{z+2}\right] + \dfrac{11}{6}\left[\dfrac{z}{z-6}\right]$ (1M)

$\therefore\ Z_T^{-1}\left[\bar{u}(z)\right] = \dfrac{1}{6} Z_T^{-1}\left[\dfrac{z}{z+2}\right] + \dfrac{11}{6} Z_T^{-1}\left[\dfrac{z}{z-4}\right]$ (1M)

ie $Z_T^{-1}\left(\bar{u}(z)\right) = \dfrac{1}{6}\left[(-2)^n + 11\,(4)^n\right]$ (1M)

(7M)

(c) Given: $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$; $u_0 = 0$, $u_1 = 1$.

Taking Z-transforms on both sides of the given equation, we have

$$Z_T\left[u_{n+2}\right] + 4Z_T\left[u_{n+1}\right] + 3Z_T\left[u_n\right] = Z_T\left[3^n\right]$$

ie $z^2\left[\bar{u}(z) - u_0 - u_1 z^{-1}\right] + 4z\left[\bar{u}(z) - u_0\right] + 3\bar{u}(z) = \dfrac{z}{z-3}$ (1M)

ie $(z^2+4z+3)\,\bar{u}(z) - z = \dfrac{z}{z-3}$

ie $(z^2+4z+3)\,\bar{u}(z) = \dfrac{z}{z-3} + z$

ie $(z^2+4z+3)\,\bar{u}(z) = \dfrac{z + z(z-3)}{(z-3)}$

$\therefore\ \bar{u}(z) = \dfrac{z + z^2 - 3z}{(z-3)(z^2+4z+3)}$

ie $\bar{u}(z) = \dfrac{z^2 + z - 3z}{(z-3)(z+3)(z+1)}$

ie $\dfrac{\bar{u}(z)}{z} = \dfrac{z+1-3}{(z-3)(z+3)(z+1)} = \dfrac{z-2}{(z-3)(z+3)(z+1)}$ (1M)

Q.No.	Solution and Scheme	Marks
	Now, let $\dfrac{z-2}{(z-3)(z+3)(z+1)} = \dfrac{A}{z-3} + \dfrac{B}{z+3} + \dfrac{C}{z+1}$	
	or $z-2 = A(z+3)(z+1) + B(z-3)(z+1) + C(z-3)(z+3)$	
	Put $z=3 \Rightarrow 1 = A(6)(4) \Rightarrow \boxed{A = \frac{1}{24}}$	(1M)
	Put $z=-3 \Rightarrow -5 = B(-6)(2) \Rightarrow \boxed{B = \frac{-5}{12}}$	(1M)
	Put $z=-1 \Rightarrow -3 = C(-4)(2) \Rightarrow \boxed{C = \frac{3}{8}}$	(1M)
	$\therefore \dfrac{\bar{u}(z)}{z} = \dfrac{1}{24(z-3)} - \dfrac{5}{12(z+3)} + \dfrac{3}{8(z+1)}$	
	or $\bar{u}(z) = \dfrac{1}{24}\left[\dfrac{z}{z-3}\right] - \dfrac{5}{12}\left[\dfrac{z}{z+3}\right] + \dfrac{3}{8}\left[\dfrac{z}{z+1}\right]$	(1M)
	$\therefore Z_T^{-1}[\bar{u}(z)] = \dfrac{1}{24} Z_T^{-1}\left[\dfrac{z}{z-3}\right] - \dfrac{5}{12} Z_T^{-1}\left[\dfrac{z}{z+3}\right] + \dfrac{3}{8} Z_T^{-1}\left[\dfrac{z}{z+1}\right]$	
	ie $u_n = \dfrac{1}{24}(3)^n - \dfrac{5}{12}(-3)^n + \dfrac{3}{8}(-1)^n$	
	$= \dfrac{(3)^n - 10(-3)^n + 9(-1)^n}{24}$	
	$\therefore u_n = \dfrac{1}{24}\left[9(-1)^n - 10(-3)^n + (3)^n\right]$	(1M)
		(7M)
Q.07 (a)	(i) Given: $u_{xx} + 4u_{xy} + 4u_{yy} - u_x + 2u_y = 0$ —— (1)	
	Comparing (1) with	
	$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + F(x,y,u,u_x,u_y) = 0$, —— (2)	(2M)
	we get $A=1$, $B=4$, $C=4$	

Q.No.	Solution and Scheme	Marks
	$\therefore B^2 - 4AC = (4)^2 - (4 \times 1 \times 4) = 0$	(2M)
	$\therefore$ The equation is parabolic.	
	(ii) $x^2 u_{xx} + (1-y^2) u_{yy} = 0, \ -1 < y < 1$	
	Comparing given equation with (2), we get	
	$A = x^2, \ B = 0, \ C = 1 - y^2$	
	$\therefore B^2 - 4AC = 0 - 4(x^2)(1-y^2) = 4x^2(y^2-1)$	
	For all 'x' between $-\infty$ and ∞, x^2 is positive	
	For all 'y' between -1 and 1, $y^2 < 1$	
	$\therefore B^2 - 4AC < 0$	(2M)
	Hence the equation is elliptic	
	(iii) $(1+x^2) u_{xx} + (5+2x^2) u_{xt} + (4+x^2) u_{tt} = 0.$	
	Comparing given equation with (2), we get	
	$A = (1+x^2), \ B = (5+2x^2), \ C = (4+x^2)$	
	$\therefore B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2)$	
	$= 25 + \cancel{4x^4} + \cancel{20x^2} - 16 - \cancel{4x^2} - \cancel{16x^2} - \cancel{4x^4}$	
	$\therefore B^2 - 4AC = 9 > 0$	(2M)
	Hence the equation is hyperbolic.	
	(iv) $y^2 u_{xx} - 2y u_{xy} + u_{yy} - u_y - 8y = 0.$	
	Comparing given equation with (2), we get	
	$A = y^2, \ B = -2y, \ C = 1$	
	$\therefore B^2 - 4AC = (-2y)^2 - 4(y^2)(1) = 0$	
	$\therefore$ The equation is parabolic.	(2M)
		(10M)

Q.No.	Solution and Scheme	Marks

(b) Here, $c^2 = 4, h = 1$ and $k = \frac{1}{8}$. Then $\alpha = \frac{c^2 k}{h^2} = \frac{1}{2}$ (1M)

$\therefore$ We have Bendre – Schmidt's recurrence relation

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right) \text{ —— (1)}$$

$\therefore u_{0,j} = 0$ and $u_{8,j} = 0$ for all values of 'j' ie the enteries in the first and last column are zero.

Since $u(x,0) = 4x - \frac{x^2}{2}$,

$$u_{i,0} = 4i - \frac{i^2}{2}$$

$= 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5$ for $i = 0,1,2,3,4,5,6,7$ (1M)

at $t = 0$.

These are the enteries of the first row.

Taking $j = 0$ in (1), we have $u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0}\right)$

Taking $i = 1, 2, 3, \ldots, 7$ successively, we get

$$u_{1,1} = \frac{1}{2}\left(u_{0,0} + u_{2,0}\right) = \frac{1}{2}(0+6) = 3$$

$$u_{2,1} = \frac{1}{2}\left(u_{1,0} + u_{3,0}\right) = \frac{1}{2}(3.5 + 7.5) = 5.5$$ (1M)

$$u_{3,1} = \frac{1}{2}\left(u_{2,0} + u_{4,0}\right) = \frac{1}{2}(6+8) = 7$$

$u_{4,1} = 7.5,\ u_{5,1} = 7,\ u_{6,1} = 5.5,\ u_{7,1} = 3.$

These are the enteries in the second row.

Putting $j = 1$ in (1), the enteries of the third row are given by:

$$u_{1,2} = \frac{1}{2}\left(u_{i-1,1} + u_{i+1,1}\right)$$ (1M)

Similarly putting $j = 2, 3, 4$ successively in (1), the enteries of the fourth, fifth and sixth rows are obtained (3M)

Hence the values of $u_{i,j}$ are as given in the following table.

Q.No.	Solution and Scheme	Marks

$j \backslash i$	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

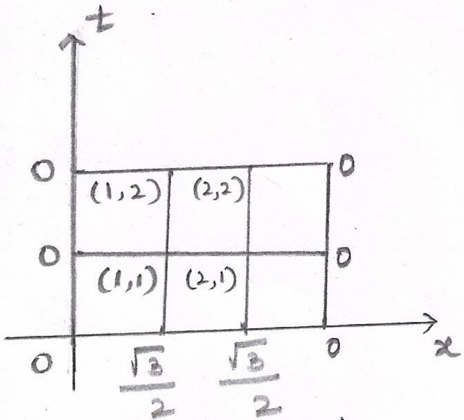
(3M)

(10M)

Q.08

(a) Here, $c^2 = 1$, $h = 1/3$, $k = 1/36$

So that $\alpha = \frac{kc^2}{h^2} = \frac{1}{4}$.



Also, $u_{1,0} = \sin(\pi/3) = \sqrt{3}/2$

$u_{2,0} = \sin(2\pi/3) = \sqrt{3}/2$ (4M)

And all boundary values are zero as shown in the figure.

By Schmidt's formula, we have

$$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha) u_{i,j} + \alpha u_{i+1,j}$$ (1M)

becomes $u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}\right]$ (1M)

For $i = 1, 2$; $j = 0$:

$$u_{1,1} = \frac{1}{4}\left[u_{0,0} + 2u_{1,0} + u_{2,0}\right] = \frac{1}{4}\left[0 + 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right] = 0.65$$ (1M)

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + 2u_{2,0} + u_{3,0}\right] = \frac{1}{4}\left[\frac{\sqrt{3}}{2} + 2 \times \frac{\sqrt{3}}{2} + 0\right] = 0.65$$ (1M)

For $i = 1, 2$; $j = 1$: (1M)

$$u_{1,2} = \frac{1}{4}\left[u_{0,1} + 2u_{1,1} + u_{2,1}\right] = 0.49$$ (1M)

$$u_{2,2} = \frac{1}{4}\left[u_{1,1} + 2u_{2,1} + u_{3,1}\right] = 0.49$$ (10M)

Q.No.	Solution and Scheme	Marks
(b)	Here, $\frac{h}{k} = c = 5$.	
	$\therefore$ The difference equation is given by:	(1M)
	$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ —— (1)	
	Which gives a Convergent solution (since $k < h$)	
	Now since $u(0,t) = u(5,t) = 0$	
	$\therefore u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of 'j'.	
	ie the enteries in the first and last Columns are zero.	
	Since $u(x,0) = \begin{cases} 20x, & 0 \le x \le 1 \\ 5(5-x), & 1 \le x \le 5 \end{cases}$, $u(i,0) = \begin{cases} 20i, & 0 \le i \le 1 \\ 5(5-i), & 1 \le i \le 5 \end{cases}$	
	$\therefore u(i,0) = 20, 15, 10, 5$ for $i = 1,2,3,4$.	(1M)
	These are the entries of the first row.	
	Also $u_t(x,0) = 0$ becomes	
	$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$ when $j=0$, giving $u_{i,1} = u_{j,-1}$ —— (2)	
	Putting $j=0$ in (1), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$	
	ie $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,1}$ [$\because$ From (2)]	
	or $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$	
	Taking $i=1,2,3,4$ respectively, we obtain	
	$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = 7.5$; $u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = 15$;	(1M)
	$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = 10$; $u_{4,1} = \frac{1}{2}(u_{3,0} + u_{5,0}) = 5$.	
	These are the entries of the second row.	
	Putting $j=1$ in (1), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$.	
	Taking $i=1,2,3,4$ respectively, we obtain	
	$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 15 - 20 = -5$	

Q.No.	Solution and Scheme	Marks

$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 7.5 + 10 - 15 = 2.5$

$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 15 + 5 - 10 = 10$ (1M)

$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 10 + 0 - 5 = 5$

These are the entries of the third row.

Putting $j=2$ in (1), $u_{i,3} = u_{i-1,2} + u_{i+1,2} - u_{i,1}$

Taking $i = 1, 2, 3, 4$ respectively, we obtain

$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 + 10 - 15 = -5$

$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = -5 + 10 - 15 = -10$ (1M)

$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 2.5 + 5 - 10 = -2.5$

$u_{4,3} = u_{3,2} + u_{5,2} - u_{4,1} = 10 + 0 - 5 = 5$

These are the entries of the fourth row.

Now the equation of the vibrating string of length 'l' is

$u_{tt} = c^2 u_{xx}$.

$\therefore$ Its period of vibration $= \frac{2l}{c} = \frac{2 \times 25}{5} = 10$ sec. (1M)

This shows that we have to compute $u(x,t)$ upto $t=5$.

Putting $j=3$ in (1), $u_{i,4} = u_{i-1,4} + u_{i+1,4} - u_{i,3}$

Taking $i = 1, 2, 3, 4$ respectively, we obtain (1M)

$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 + (-10) - (-5) = -10 + 5 = -5$

$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = -5 + 5 - 10 = -10$

$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -10 + 5 - 10 = -15$

$u_{4,4} = u_{3,3} + u_{5,4} - u_{4,2} = -2.5 + 0 - 5 = -7.5$

These are the entries of the fifth row.

The values of $u_{i,j}$ are as shown in the following table. [The entries of the sixth row are obtained IIIly as above] (1M)

j \ i	0	1	2	3	4	5
0	0	20	15	10	5	0
1	0	7.5	15	10	5	0
2	0	-5	2.5	10	5	0
3	0	-5	-10	-2.5	5	0
4	0	-5	-10	-15	-7.5	0
5	0	-5	-10	-15	-20	0

(2M)

(10M)

Q.No.	Solution and Scheme	Marks
Q.09		
(a)	Given: $\dfrac{d^2y}{dx^2} = y + x\dfrac{dy}{dx}$; $y(0)=1, y'(0)=0$.	
	ie $\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - y = 0$; $y(0)=1$, $y'(0)=0$ —— (1)	
	Here, $x_0=0$, $y_0=1$, $y_0'=0$, $x_1=0.2$	
	Step length $h = x_1 - x_0 = 0.2 - 0 = 0.2$	
	Put $\dfrac{dy}{dx} = y' = z = f(x,y,z)$	
	$\therefore \dfrac{d^2y}{dx^2} = y'' = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{dz}{dx}$.	(1M)
	Therefore, from (1),	
	$\dfrac{dz}{dx} - xz - y = 0$	
	ie $\dfrac{dz}{dx} = y + xz = g(x,y,z)$	
	Now, $k_1 = hf(x_0, y_0, z_0)$	(1M)
	$= 0.2f(0,1,0)$	
	$= 0.2 \times 0$	
	$\therefore k_1 = 0$	
	Now, $l_1 = hg(x_0, y_0, z_0)$	
	$= 0.2 \times g(0,1,0)$	
	$= 0.2(1+0)$	
	$\therefore l_1 = 0.2$	
	Now, $k_2 = hf\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{l_1}{2}\right)$	(1M)

Q.No.	Solution and Scheme	Marks
	$\therefore k_2 = 0.2 f\left(0+\frac{0.2}{2}, 1+\frac{0}{2}, 0+\frac{0.2}{2}\right)$ $= 0.2 f(0.1, 1, 0.1)$ $= 0.2 \times 0.1$ $\therefore k_2 = 0.02$ Now, $l_2 = hg\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$ $= 0.2 g\left(0+\frac{0.2}{2}, 1+\frac{0}{2}, 0+\frac{0.2}{2}\right)$ $= 0.2 g(0.1, 1, 0.1)$ $= 0.2\left[1+(0.1)(0.1)\right]$ $\therefore l_2 = 0.202$	
	Now, $k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$ $= 0.2 f\left(0+\frac{0.2}{2}, 1+\frac{0.02}{2}, 0+\frac{0.202}{2}\right)$ $= 0.2 f(0.1, 1.01, 1.101)$ $= 0.2 \times 0.101$ $\therefore k_3 = 0.0202$ Now, $l_3 = hg\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$ $= 0.2 g(0.1, 1.01, 0.101)$ $= 0.2\left[1.01+(0.1)(0.101)\right]$ $= 0.2040$ Now, $k_4 = hf(x_0+h, y_0+k_3, z_0+l_3)$ $= 0.2 f(0+0.2, 1+0.0202, 0+0.2040)$ $\therefore k_4 = 0.0408$	(1M)

Q.No.	Solution and Scheme	Marks
	Now, $l_4 = hg(x_0+h, y_0+k_3, z_0+l_3)$	
	$= 0.2g(0.2, 1.0202, 0.2040)$	
	$= 0.2[1.0202+(0.2)(0.2040)]$	
	$\therefore l_4 = 0.2122$	
	Now, $k = \frac{1}{6}[k_1+2k_2+2k_3+k_4]$	(1M)
	$= \frac{1}{6}[0+2(0.02)+2(0.0202)+0.0408]$	
	$\therefore k = 0.0202.$	
	Now, $L = \frac{1}{6}[l_1+2l_2+2l_3+l_4]$	
	$= \frac{1}{6}[0.2+(2)(0.202)+(2)(0.2040)+0.2122]$	
	$\therefore L = 0.2040$	
	Now, $y_1 = y(x_1) = y(0.2) = y_0+k$	
	ie $y_1 = 1+0.0202$	
	ie $y_1 = 1.0202$	(1M)
	or $\underline{\underline{y(0.2) = 1.0202}}$.	(6M)
(b)	Let $f(x,y,y') = y^2+(y')^2+2ye^x$	(1M)
	Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,	
	$(2y+2e^x) - \frac{d}{dx}(2y') = 0$ or $y+e^x-y''=0.$	(1M)
	ie $y''-y=e^x$ or $(D^2-1)y=e^x$ Where $D=\frac{d}{dx}$	
	AE is $m^2-1=0$ $\therefore m=\pm 1$	(1M)

Q.No.	Solution and Scheme	Marks

Hence, $CF = y_c = C_1 e^{x} + C_2 e^{-x}$ (1M)

$$PI = y_p = \frac{e^{x}}{D^2-1} = \frac{e^{x}}{0}$$, on replaying D by 1. (1M)

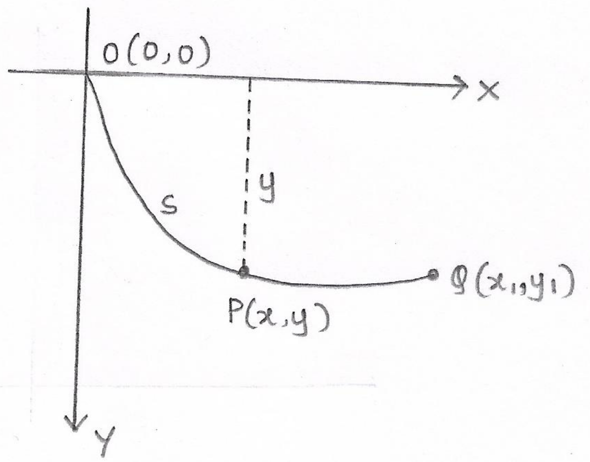
$$y_p = x\frac{e^{x}}{2D} = \frac{xe^{x}}{2}$$

We have, $y = y_c + y_p$ (1M)

Thus, $y = C_1 e^{x} + C_2 e^{-x} + \frac{xe^{x}}{2}$ (1M)

(7M)

(c)



Let the particle start from the point O (initially at rest) and slide along the curve OQ.

Let the particle be at P(x,y) at any time 't' and let OP = s be the arc length (1M)

We know that from the principle of work and energy the work done in moving the particle from O to P is equal to the difference between the kinetic energy at 'P' and at 'O'

ie $mgy = \frac{1}{2}mv^2 - 0$ But, $v = \frac{ds}{dt}$

$$\therefore mgy = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 \text{ or } \frac{ds}{dt} = \sqrt{2gy}$$ (1M)

This is the velocity of the particle at the point P(x,y). Hence the time T required by the particle to move from O to Q is given by

$$T = \int_0^T dt = \int_0^{x_1} \frac{dt}{dx}\,dx = \int_0^{x_1} \frac{dt}{ds}\cdot\frac{ds}{dx}\,dx$$

Using, $\frac{dt}{ds} = \frac{1}{\sqrt{2gy}}$ and $\frac{ds}{dx} = \sqrt{1+y'^2}$ we have,

Q.No.	Solution and Scheme	Marks

$$T = \int_0^{x_1} \frac{1}{\sqrt{2gy}} \sqrt{1+y'^2}\, dx$$

ie $$T = \frac{1}{\sqrt{2g}} \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx$$ (1M)

We need to find $y(x)$ such that 'T' is minimum.

Let $f(x,y,y') = \dfrac{\sqrt{1+y'^2}}{\sqrt{y}}$ which is independent of 'x'.

Hence we can take Euler's equation in the form,

$$f - y'\frac{\partial f}{\partial y'} = \text{Constant} = c$$

ie $$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - y'\left\{\frac{1}{\sqrt{y}} \frac{1}{2\sqrt{1+y'^2}} \cdot 2y'\right\} = c$$

ie $$\frac{1}{\sqrt{y}\sqrt{1+y'^2}}\left\{1+y'^2 - y'^2\right\} = c$$

or $$\frac{1}{\sqrt{y}\sqrt{1+y'^2}} = c$$

ie $$\sqrt{y}\sqrt{1+y'^2} = \frac{1}{c} = \sqrt{a} \text{ (say)}$$

By squaring, $y(1+y'^2) = a$ or $y'^2 = \dfrac{a}{y} - 1$ or $y'^2 = \dfrac{a-y}{y}$

ie $$\frac{dy}{dx} = \sqrt{\frac{a-y}{y}}$$

or $$dx = \sqrt{\frac{y}{a-y}}\, dy$$

$$\therefore \int dx = \int \sqrt{\frac{y}{a-y}}\, dy$$

Q.No.	Solution and Scheme	Marks
	ie $x = \int \sqrt{\frac{y}{a-y}}\, dy + b$	(1M)
	Put $y = a\sin^2(\theta/2)$ $\therefore dy = a \cdot 2\sin(\theta/2)\cos(\theta/2) \cdot \frac{1}{2}$	
	$\therefore x = \int \frac{\sqrt{a}\sin(\theta/2)}{\sqrt{a}\cos(\theta/2)} \cdot a\sin(\theta/2)\cos(\theta/2)\, d\theta + b$	
	ie $x = a\int \sin^2(\theta/2)\, d\theta + b$	
	ie $x = a\int \frac{1-\cos\theta}{2}\, d\theta + b$	
	ie $x = \frac{a}{2}(\theta - \sin\theta) + b$	(1M)
	Also we have when $\theta = 0, y = 0$ and we must have $x = 0$ also since $(0,0)$ is a point on the curve. This gives $b = 0$.	
	Hence, $x = \frac{a}{2}(\theta - \sin\theta)$; $y = a\sin^2\left(\frac{\theta}{2}\right) = \frac{a}{2}(1-\cos\theta)$	
	Taking, $a/2 = k$ we have $x = k(\theta - \sin\theta),\ y = k(1-\cos\theta)$	
	We have obtained the required equation of the curve in the parametric form which is the equation of a cycloid.	(1M)
	Thus we can say that the path of a particle moving in a vertical plane under the action of gravity in least time is a cycloid.	(1M)
		(7M)
10 (a)	Putting $y' = \frac{dy}{dx} = z$, we obtain $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}$	
	This equation becomes, $\frac{dz}{dx} = 1 + z$ or $z' = 1 + z$.	(1M)

Q.No.	Solution and Scheme	Marks

Further, $z' = 1+z$ will give us the following values.

$z'(0) = 1+z(0) = 1+1 = 2$

$z'(0.1) = 1+z(0.1) = 2.2103$

$z'(0.2) = 1+z(0.2) = 2.4427$

$z'(0.3) = 1+z(0.3) = 2.699$

Now we tabulate these values.

x	$x_0=0$	$x_1=0.1$	$x_2=0.2$	$x_3=0.3$
y	$y_0=1$	$y_1=1.1103$	$y_2=1.2427$	$y_3=1.399$
$y'=z$	$z_0=1$	$z_1=1.2103$	$z_2=1.4427$	$z_3=1.699$
$y''=z'$	$z_0'=2$	$z_1'=2.2103$	$z_2'=2.4427$	$z_3'=2.699$

(1M)

We first Consider Milne's predictor formulae:

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3),$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

Hence, $$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(1.2103) - 1.4427 + 2(1.699)\right]$$

$$z_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(2.2103) - 2.4427 + 2(2.699)\right]$$ (1M)

$\therefore\ y_4^{(P)} = 1.5835$ and $z_4^{(P)} = 1.9835$.

Next we Consider Milne's corrector formulae:

$$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$ (1M)

$$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have, $z_4' = 1 + z_4^{(P)} = 1 + 1.9835 = 2.9835$.

Q.No.	Solution and Scheme	Marks
	Hence, $y_4^{(c)} = 1.2427 + \frac{0.1}{3}\left[1.4427 + 4(1.699) + 1.9835\right]$	
	$z_4^{(c)} = 1.4427 + \frac{0.1}{3}\left[2.4427 + 4(2.699) + 2.9835\right]$	(1M)
	$\therefore\ y_4^{(c)} = 1.58344$ and $z_4^{(c)} = 1.98344$	
	Applying the corrector formula again for y_4 we obtain $y_4^{(c)} = 1.583438$.	(1M)
	Thus the required, $y(0.4) = 1.5834$	(6M)
(b)	Let $f(x,y,y') = \frac{(y')^2}{x^3}$	(1M)
	Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes,	
	$0 - \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 0$	
	ie $\frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 0$	(1M)
	or $\frac{2y'}{x^3} = C_1$	
	ie $2y' = C_1 x^3$	
	or $2\frac{dy}{dx} = C_1 x^3$	
	ie $2\,dy = C_1 x^3\,dx$	(1M)
	ie $2\int dy = C_1 \int x^3\,dx$	
	or $2y = \frac{C_1 x^4}{4} + C_2$	(2M)
	or $y = \frac{x^4}{4} C_1 + C_2$	(2M)
		(7M)

Q.No.	Solution and Scheme	Marks

(c) Let $f(x,y,y') = (y')^2 + 12xy$ (1M)

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes: (1M)

$$12x - \frac{d}{dx}(2y') = 0$$ (1M)

ie $6x - \frac{d}{dx}(y') = 0$

or $6x - y'' = 0$

ie $y'' = 6x$

ie $\int y'' dx = 6\int x$

ie $y' = \frac{6x^2}{2} + c_1$

ie $y' = 3x^2 + c_1$

ie $\int y' dx = 3\int x^2 dx + \int c_1 dx$

or $y = x^3 + c_1 x + c_2$ (1M)

By data, $y(0) = 0$ and $y\left(\frac{\pi}{2}\right) = 0$ (1M)

$\therefore 0 = 0 + 0 + c_2$ and $0 = \left(\frac{\pi}{2}\right)^3 + c_1\left(\frac{\pi}{2}\right) + 0$

ie $c_2 = 0$

ie $0 = \frac{\pi^3}{8} + \frac{\pi c_1}{2}$

or $\frac{\pi c_1}{2} = \frac{-\pi^3}{8}$

ie $c_1 = \frac{-\pi^2}{4}$ (1M)

Therefore, $y = x^3 - \frac{\pi^2 x}{4}$ (1M)

(7M)

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