CBCS SCHEME

USN 2VD23CS078

BMATS101

First Semester B.E./B.Tech. Degree Examination, Dec.2023/Jan.2024
Mathematics – I for CSE Stream

Time: 3 hrs. Max. Marks: 100

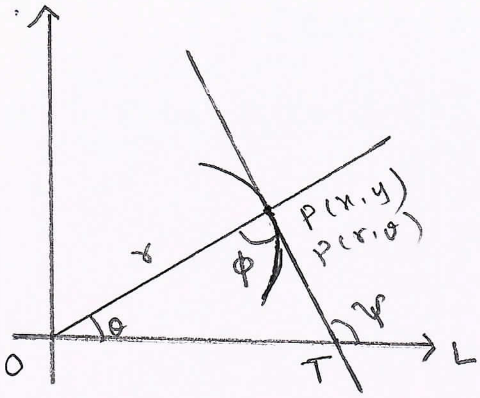
***Note:** 1. Answer any FIVE full questions, choosing ONE full question from each module.*
2. VTU Formula Hand Book is permitted.
3. M : Marks , L: Bloom's level , C: Course outcomes.

		Module – 1	M	L	C
Q.1	a.	With usual notation prove that $\tan\phi = r\cdot\frac{d\theta}{dr}$.	6	L2	CO1
	b.	Find the angle between the curves $r = 6\cos\theta$ and $r = 2(1+\cos\theta)$.	7	L2	CO1
	c.	Find the radius of curvature for the Folium of De-Cartes $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on it.	7	L2	CO1
		OR			
Q.2	a.	Show that the curves $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$ cut each other orthogonally.	8	L2	CO1
	b.	Find the pedal equation of $r^n = a(1+\cos n\theta)$.	7	L2	CO1
	c.	Using modern mathematical tool, write a program/code to plot the curve sine and cosine curve.	5	L3	CO5
		Module – 2			
Q.3	a.	Using Maclaurin's series, expand $\sqrt{1+\sin 2x}$ in powers of x upto the terms x^4.	7	L2	CO1
	b.	If $U = e^{ax+by} f(ax - by)$, prove that $b\frac{\partial U}{\partial x} + a\frac{\partial U}{\partial y} = 2abU$.	6	L2	CO1
	c.	Find the extreme values of the function $\sin x + \sin y + \sin(x+y)$.	7	L3	CO1
		OR			
Q.4	a.	Evaluate the $\lim_{x\to 0}\left[\frac{a^x + b^x + c^x + d^x}{4}\right]^{1/x}$.	8	L2	CO1
	b.	If $U = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial U}{\partial x} + \frac{1}{3}\frac{\partial U}{\partial y} + \frac{1}{4}\frac{\partial U}{\partial z} = 0$.	7	L2	CO1
	c.	Using modern mathematical tool, write a program/code to evaluate $\underset{x\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{x}\right)^x$.	5	L3	CO5
		Module – 3			
Q.5	a.	Solve $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$.	6	L2	CO2
	b.	Find the orthogonal trajectories of the family $r = a(1 + \sin\theta)$.	7	L3	CO2
	c.	Find the solution of the equation $x^2(y - Px) = P^2y$ by reducing into Clairaut's form using the substitution $X = x^2$, $Y = y^2$.	7	L2	CO2
		OR			

Q.6	a.	Solve $(8xy - 9y^2)dx + 2(x^2 - 3xy)dy = 0$.	6	L2	CO2
	b.	A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L and resistant R. Find the current at any time t given that the current is initially zero when $t = 0$.	7	L3	CO2
	c.	Solve $x(y')^2 - (2x + 3y)y' + 6y = 0$.	7	L2	CO2
		Module – 4			
Q.7	a.	i) Find the last digit in 7^{289}. ii) Find the remainder when $135 \times 74 \times 48$ is divided by 7.	7	L2	CO3
	b.	Solve the linear congruence $6x \equiv 15 \pmod{21}$.	6	L2	CO3
	c.	Using Wilson's theorem, show that $4(29)! + 5!$ is divisible by 31.	7	L2	CO3
		OR			
Q.8	a.	Solve the set of simultaneous congruences $x \equiv 5 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv 1 \pmod{11}$.	7	L2	CO3
	b.	Solve $7x + 3y \equiv 10 \pmod{16}$, $2x + 5y \equiv 9 \pmod{16}$.	6	L2	CO3
	c.	Show that $2^{340} - 1$ is divisible by 31, using Fermat's little theorem.	7	L2	CO3
		Module – 5			
Q.9	a.	Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix}$	6	L2	CO4
	b.	Solve the system of equation by using Gauss-Jordan method. $x + y + z = 8$, $-x - y + 2z = -4$, $3x + 5y - 7z = -14$	7	L3	CO4
	c.	Using Rayleigh's power method, find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ by taking initial vector as $[1\ \ 1\ \ 1]^T$. Perform 6 iterations.	7	L3	CO4
		OR			
Q.10	a.	Solve the system of equation by using Gauss elimination method. $x + 2y + z = 3$, $2x + 3y + 3z = 10$, $3x - y + 2z = 13$	8	L3	CO4
	b.	Solve the following system of equations by Gauss-Seidal method $20x + y - 2y = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$	7	L3	CO4
	c.	Using modern mathematical tool, write a programme/code to test the consistency of the equation: $x + 2y - z = 1$, $2x + y + 4z = 2$, $3x + 3y + 4z = 1$	5	L3	CO5

* * * * *

1 a)



(1m)

Let us consider the polar curve $r=f(\theta)$
let $P(r,\theta)$ be any point on the curve
let OL be the initial line. 'θ' be the angle betⁿ radius vector and initial line
let us draw a tangent PT at the point P which makes an angle ψ with the initial line. let ϕ be angle betⁿ radius vector and tangent. wkt an exterior angle is equals to the sum of the interior opposite angles. From the figure. (1m)

$\psi = \phi + \theta \Rightarrow \tan\psi = \tan(\phi+\theta)$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1-\tan\theta\tan\phi} \quad \text{--- (1)}$$ (1m)

let (x,y) be the cartesian coordinates of pt P
∴ we have $x = r\cos\theta \quad y = r\sin\theta$
w.k.t $\tan\psi = \frac{dy}{dx}$ (slope of tangent) (1m)

$$\tan\psi = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{d}{d\theta}(r\sin\theta) \Big/ \frac{d}{d\theta}(r\cos\theta)$$

$$= \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta} \quad \text{where } r' = \frac{dr}{d\theta}$$

Divide both NR and DR by $r'\cos\theta$ we have

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} \quad \text{--- (2)}$$ (1m)

Comparing (1) and (2) $\tan\phi = \frac{r}{r'} = \frac{r}{\frac{dr}{d\theta}}$ (1m)

$\tan\phi = r \cdot \frac{d\theta}{dr}$

1 b)	$r = 6\cos\theta \qquad r = 2(1+\cos\theta)$	
	Taking log on both sides diff w.r.t θ.	
	$\log r = \log 6 + \log\cos\theta$	
	$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{\cos\theta} \Rightarrow \cot\phi_1 = -\tan\theta$	
	$\cot\phi_1 = \cot(\pi/2+\theta) \Rightarrow \phi_1 = \pi/2+\theta$.	(2m)
	$\log r = \log 2 + \log(1+\cos\theta)$	
	$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = -\frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\cos^2\theta/2}$	
	$\cot\phi_2 = -\tan\theta/2 \Rightarrow \cot\phi_2 = \cot(\pi/2+\theta/2)$	
	Angle of intersection $= \lvert\phi_1-\phi_2\rvert$ $\Rightarrow \phi_2 = \pi/2+\theta/2$	(2m)
	$= \lvert \pi/2+\theta-\pi/2-\theta/2\rvert = \lvert\theta/2\rvert = \theta/2$	(1m)
	comparing RHS of given eqn	
	$\overset{3}{\cancel{6}}\cos\theta = \cancel{2}(1+\cos\theta) \Rightarrow 3\cos\theta = 1+\cos\theta$	
	$\Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \cos^{-1}\{\frac{1}{2}\}$	
	$\theta = \frac{\pi}{3} = 60^\circ$	(1m)
	$\therefore$ Angle of intersection $= \theta/2 = \pi/6 = 30^\circ$	(1m)
1 c)	$x^3+y^3 = 3axy$ at $(3a/2,\ 3a/2)$	
	$3x^2+3y^2\frac{dy}{dx} = 3a\{x\frac{dy}{dx}+y\}$	
	$x^2+y^2y_1 = a\{xy_1+y\}$	
	$x^2+y^2y_1 = axy_1+ay$	
	$(y^2-ax)y_1 = ay-x^2$	
	$y_1 = \frac{ay-x^2}{y^2-ax}$ at $(\frac{3a}{2}, \frac{3a}{2})$	
	$y_1 = \frac{3a^2/2-\frac{9a^2}{4}}{9a^2/4-3a^2/2} = \frac{\cancel{(3a^2/2-9a^2/4)}}{-\cancel{(3a^2/2-9a^2/4)}} = -1$	
	$y_1(\frac{3a}{2}\ \frac{3a}{2}) = -1$	(2m)

$$y_2 = \frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$$

$$y_2(3a/_2\ 3a/_2) = \frac{3a^2/_4 \cdot (-4a) - (-3a^2/_4)(-4a)}{(3a^2/4)^2}$$

$$= -2 \cdot \cancel{4}a \cdot \frac{\cancel{3}a^2}{\cancel{4}} \cdot \frac{16}{3\cancel{9}a^4} = -\frac{32}{3a}$$

$$y_2(3a/_2\ 3a/_2) = -\frac{32}{3a}$$ (2 m)

we have $\rho = \frac{\{1+y_1^2\}^{3/2}}{y_2} = \frac{\{1+1\}^{3/2}}{-32/3a} = \frac{2^{3/2}}{-32/3a}$

$$\rho = \frac{-3a \cdot 2\sqrt{2}}{2^5} = \frac{-3a\,\cancel{2}\,\sqrt{2}}{\cancel{2}(\sqrt{2})^{2} \cdot 2^3} = -\frac{3a}{8\sqrt{2}}$$ (2 m)

$|\rho| = \left|-\frac{3a}{8\sqrt{2}}\right| \Rightarrow |\rho| = \frac{3a}{8\sqrt{2}}$ (1 m)

2a) $r = a(1+\sin\theta)$ $\quad r = a(1-\sin\theta)$

Taking log on both sides, diff w.r.t θ

$\log r = \log a + \log(1+\sin\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{0+\cos\theta}{1+\sin\theta} \Rightarrow \cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$ (2 m)

$\log r = \log a + \log(1-\sin\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{0-\cos\theta}{1-\sin\theta} \Rightarrow \cot\phi_2 = -\frac{\cos\theta}{1-\sin\theta}$ (2 m)

$$\cot\phi_1 \cdot \cot\phi_2 = \left(\frac{\cos\theta}{1+\sin\theta}\right)\left(-\left[\frac{\cos\theta}{1-\sin\theta}\right]\right)$$

$$= -\frac{\cos^2\theta}{1-\sin^2\theta} = -\frac{\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}} = -1$$ (2 m)

$\cot\phi_1 \cdot \cot\phi_2 = -1$

$\Rightarrow$ The two curves intersect orthogonally (2 m)

2b) $r^n = a(1+\cos n\theta)$ —— (1)

Taking log on both sides, diff w.r.t θ

$n\log r = \log a + \log(1+\cos n\theta)$

$n\frac{1}{r}\frac{dr}{d\theta} = \frac{0 - \sin n\theta \cdot n}{1+\cos n\theta} \Rightarrow \cot\phi = -\frac{\cancel{2}\sin n\theta/_2 \cos n\theta/_2}{\cancel{2}\cos^2 n\theta/_2}$

$\cot\phi = -\tan n\theta/_2 \Rightarrow \cot\phi = \cot(\pi/_2 + n\theta/_2)$

$\phi = n\theta/_2 + \pi/_2$ (2m)

consider $p = r\sin\phi \Rightarrow p = r\sin(\pi/_2 + n\theta/_2)$

$p = r\cos n\theta/_2$ —— (2) (2m)

From (1) $r^n = a\, 2\cos^2 n\theta/_2 \Rightarrow \cos^2 n\theta/_2 = \frac{r^n}{2a}$ —— (3)

squaring Eqn (2) we get (1m)

$p^2 = r^2\cos^2 n\theta/_2$ —— (4)

Using eqn (3) in eqn (4)

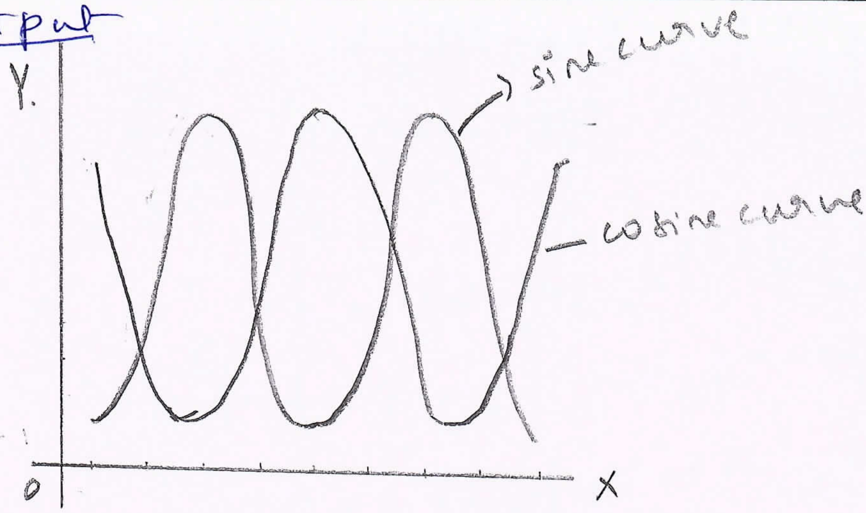
$p^2 = r^2 \cdot \frac{r^n}{2a} \Rightarrow 2ap^2 = r^{n+2}$ (2m)

which is required pedal eqn

2c)

```
Import numpy as np
Import matplotlib.pyplot as plt
x = np.arange(-10, 10, 0.001)
y1 = np.sin(x)
y2 = np.cos(x)
Plot plt.plot(x, y1, x, y2)                                   (4m)
plt.title("sine curve and cosine curve)
plt.xlabel("values of x")
plt.ylabel("values of sin(x) and cos(x)")
plt.grid()
show plt.show()
```

Output



(1m)

3a) $f(x) = \sqrt{1+\sin 2x} = \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}$

$f(x) = \sqrt{(\sin x + \cos x)^2} = \sin x + \cos x$ (1m)

$f(x) = \sin x + \cos x$ $\quad f(0) = 0 + 1 = 1$

$f'(x) = \cos x - \sin x$ $\quad f'(0) = 1 - 0 = 1$

$f''(x) = -\sin x - \cos x$ $\quad f''(0) = -0 - 1 = -1$

$f'''(x) = -\cos x + \sin x$ $\quad f'''(0) = -1 + 0 = -1$

$f^{IV}(x) = \sin x + \cos x$ $\quad f^{IV}(0) = 0 + 1 = 1$

$f^{V}(x) = \cos x - \sin x$ $\quad f^{V}(0) = 1 - 0 = 1$ (4m)

$f^{IV}(x) = -\sin x - \cos x$ $\quad f^{IV}(0) = -0 - 1 = -1$

we have maclaurin's series expansion

$$f(x) = f(0) + \frac{f'(0) \cdot x}{1!} + \frac{x^2}{2!} f''(0) + \cdots$$

$$f(x) = 1 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(-1) + \frac{x^4}{24}(1) + \frac{x^5}{120}(1) + \frac{x^6}{720}(-1) + \cdots$$

$$\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720} + \cdots$$ (1m)

3 b)	$u = e^{ax+by} f(ax-by)$	
	$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by) \cdot a + f(ax-by) \cdot e^{ax+by} \cdot a$	
	$\frac{\partial u}{\partial x} = a e^{ax+by} f'(ax-by) + au$	(2m)
	$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by) e^{ax+by} \cdot b$	
	$\frac{\partial u}{\partial y} = -b e^{ax+by} f'(ax-by) + bu$	(2m)
	$b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = ab e^{ax+by} \cancel{f'(ax-by)} + abu - ab e^{ax+by} \cancel{f'(ax-by)} + abu$	2m
	$b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$	(2m)
3c)	$f(x,y) = \sin x + \sin y + \sin(x+y)$	
	$f_x = \cos x + \cos(x+y)$, $f_y = \cos y + \cos(x+y)$	
	$f_x = f_y = 0 \Rightarrow \cos x + \cos(x+y) = 0$, $\cos y + \cos(x+y) = 0$	
	$\cos(x+y) = -\cos x$ $\quad \cos(x+y) = -\cos y$	
	$\Rightarrow -\cos x = -\cos y \Rightarrow x = y$	
	Put $y = x$ in $\cos x + \cos(x+y) = 0$	
	$\Rightarrow \cos x + \cos 2x = 0 \Rightarrow \cos x + 2\cos^2 x - 1 = 0$	
	$\Rightarrow 2\cos^2 x + \cos x - 1 = 0$	
	$\Rightarrow (2\cos x - 1)(\cos x + 1) = 0$	
	$\cos x = \frac{1}{2}$ $\quad \cos x = -1 \Rightarrow x = \pi/3,\ x = \pi$	
	$y = x \Rightarrow y = \pi/3 \quad y = \pi$	
	Hence $(\pi/3, \pi/3)$ (π, π) are stationary points	(3m)
	$f_{xx} = A = -\sin x - \sin(x+y)$	
	$f_{yy} = C = -\sin y - \sin(x+y)$	
	$f_{xy} = B = -\sin(x+y)$	(2m)

At $(\pi/3, \pi/3)$

$A = -\sin\pi/3 - \sin(2\pi/3) = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

$B = -\sqrt{3}/2$

$C = -\sqrt{3}$

$AC - B^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0 \quad , \quad A = -\sqrt{3} < 0$

Hence $f(x, y)$ is maximum at $(\pi/3, \pi/3)$ and maximum value is

$f(\pi/3, \pi/3) = \sin\pi/3 + \sin\pi/3 + \sin\frac{2\pi}{3}$

$= \sqrt{3}/2 + \sqrt{3}/2 + \sqrt{3}/2 = 3\sqrt{3}/2$ (2m)

At (π, π)

$A = 0 \;,\; B = 0 \quad C = 0$

The case need further investigation

4 a) $k = \lim_{x\to 0}\left\{\frac{a^x + b^x + c^x + d^x}{4}\right\}^{1/x}$; 1^∞ (1m)

$\log_e k = \lim_{x\to 0}\frac{1}{x}\cdot\log_e\left(\frac{a^x + b^x + c^x + d^x}{4}\right)$

$= \lim_{x\to 0}\frac{\log_e \frac{a^x + b^x + c^x + d^x}{4}}{x} \quad \left(\frac{0}{0}\right)$ (1m)

Applying L Hospital's rule

$\log_e k = \lim_{x\to 0}\frac{1\,[a^x\log a + b^x\log b + c^x\log c + d^x\log d]/4}{a^x + b^x + c^x + d^x/4}$ (2m)

$= \lim_{x\to 0}\frac{\cancel{4}}{a^x + b^x + c^x + d^x}\cdot\frac{1}{\cancel{4}}\cdot[a^x\log a + b^x\log b + c^x\log c + d^x\log d]$

$= \frac{\log a + \log b + \log c + \log d}{1+1+1+1} = \frac{1}{4}\log(abcd)$ (2m)

$\log_e k = \log(abcd)^{1/4} \Rightarrow k = (abcd)^{1/4}$ (2m)

4b) $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

$p = 2x-3y \quad q = 3y-4z \quad r = 4z-2x$ (1m)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}2 + 0 + \frac{\partial u}{\partial r}(-2) = 2\left(\frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}\right)$$ (1m)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-3) + \frac{\partial u}{\partial q}3 + 0 = 3\left(\frac{\partial u}{\partial q} - \frac{\partial u}{\partial p}\right)$$ (1m)

$$\frac{\partial u}{\partial z} = 0 + \frac{\partial u}{\partial q}(-4) + \frac{\partial u}{\partial r}(4) = 4\left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial q}\right)$$ (1m)

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial q}}$$ (2m)

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$$ (1m)

4c)
```
From sympy import *
from math import inf
x = symbol('x')
l = lim((1+1/x)**x, x, inf).doit()
display(l)
```
(4m)

Output - e

(1m)

5a) $$\frac{dy}{dx} + \frac{y}{x} = x^2y^6$$

This is Bernoulli's D.E. Divide the given eqn by y^6

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{y^5}\frac{1}{x} = x^2$$

Put $\frac{1}{y^5} = t \Rightarrow -\frac{5}{y^6}\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$$

$$\Rightarrow -\frac{1}{5}\frac{dt}{dx} + \frac{t}{x} = x^2 \Rightarrow \frac{dt}{dx} - 5\frac{t}{x} = -5x^2$$ (3m)

$P = -5/x \quad Q = -5x^2$

$$IF = e^{-5\int \frac{1}{x}dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5}$$

Hence soln is

$$t \cdot \frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5}\, dx + c$$

$$\frac{t}{x^5} = -5\int x^{-3}\,dx + c \Rightarrow \frac{t}{x^5} = -5 \cdot \frac{x^{-2}}{-2} + c$$

$$\frac{t}{x^5} = \frac{5}{2}x^2 = c \Rightarrow \frac{1}{y^5 x^5} - \frac{5}{2}\cdot\frac{1}{x^2} = c$$ (3m)

//.

5b) $r = a(1+\sin\theta)$

Taking log on bothsides diff w.r.t θ

$\log r = \log a + \log(1+\sin\theta)$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\sin\theta}(\cos\theta)$$ (1m)

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta} \quad \text{Put } \frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$$ (1m)

$$\frac{1}{\cancel{r}}\left(-r^{\cancel{2}}\frac{d\theta}{dr}\right) = \frac{\cos\theta}{1+\sin\theta} \Rightarrow \frac{1+\sin\theta}{\cos\theta}\,d\theta = -\frac{dr}{r}$$ (1m)

$$\int \frac{1}{r}dr + \int \frac{1+\sin\theta}{\cos\theta}\,d\theta = c$$

$$\int \frac{1}{r}dr + \int \sec\theta\, d\theta + \int \tan\theta\, d\theta = c$$

$$\log r + \log(\sec\theta + \tan\theta) + \log\sec\theta = \log b$$

$$\log[r(\sec\theta + \tan\theta)\cdot\sec\theta] = \log b$$

$$r\left[\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right]\frac{1}{\cos\theta} = b$$

$$r\frac{(1+\sin\theta)}{\cos^2\theta} = b, \quad \frac{r(1+\sin\theta)}{1-\sin^2\theta} = b$$

$$\frac{r(\cancel{1+\sin\theta})}{(\cancel{1+\sin\theta})(1-\sin\theta)} = b$$ (4m)

$r = b(1-\sin\theta)$ which required O.T

5c) $X = x^2 \Rightarrow \frac{dX}{dx} = 2x \qquad Y = y^2 \Rightarrow \frac{dY}{dy} = 2y$

$p = \frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\frac{dX}{dx} \Rightarrow p = \frac{1}{2y} P \, 2x = \frac{x}{y} P$

where $P = \frac{dY}{dX} \Rightarrow p = \frac{\sqrt{X}}{\sqrt{Y}} P$ (2m)

The given eqn $x^2(y - px) = p^2 y$ becomes

$$X\left(\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}} P \sqrt{X}\right) = \frac{X}{Y} P^2 \sqrt{Y}$$

$$\frac{X(Y - XP)}{\cancel{\sqrt{Y}}} = \frac{XP^2}{\cancel{\sqrt{Y}}} \Rightarrow X(Y - XP) = XP^2$$

$\Rightarrow Y - XP = P^2 \Rightarrow Y = XP + P^2$ (4m)

which is in clairaut's form

The associated general solution is

$Y = cX + c^2$. Put $X = x^2 \quad Y = y^2$

$\Rightarrow y^2 = cx^2 + c^2$

(1m)

6a) $(8xy - 9y^2)dx + 2(x^2 - 3xy)dy = 0$

$M = 8xy - 9y^2 \qquad N = 2x^2 - 6xy = 2x(x - 3y)$

$\frac{\partial M}{\partial y} = 8x - 18y \qquad \frac{\partial N}{\partial x} = 4x - 6y$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8x - 18y - 4x + 6y = 4x - 12y$

$= 4(x - 3y)$ is near to N

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{\cancel{2}x(\cancel{x-3y})} \cdot \overset{2}{\cancel{4}}(\cancel{x-3y}) = \frac{2}{x} = f(x)$$ (2m)

$IF = e^{\int f(x)\,dx} = e^{\int 2/x\,dx} = e^{2\log x} = e^{\log x^2} = x^2$ (1m)

Multiplying the given eqn by x^2

$M = 8x^3y - 9x^2y^2 \qquad N = 2x^4 - 6x^3y$

$\frac{\partial M}{\partial y} = 8x^3 - 18x^2y \qquad \frac{\partial N}{\partial x} = 8x^3 - 18x^2y$ (1m)

soln is $\int M\,dx + \int N(y)\,dy = c$ (1m)

$\int (8x^3y - 9x^2y^2)\,dx + \int 0\,dy = c$

$\cancel{8}^{2} y \frac{x^4}{\cancel{4}} - \cancel{9}^{3} y^2 \frac{x^3}{\cancel{3}} = c \Rightarrow 2x^4y - 3x^3y^2 = c$ // (2m)

5 b) The D.E for L.R circuit is $L\frac{di}{dt} + Ri = E(t)$ (1m)

$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at} \quad \because E(t) = Ee^{-at}$

soln is $(i)\,e^{R/L\,t} = \int \frac{E}{L} e^{-at} e^{R/L\,t}\,dt + c$ (1m)

$i\,e^{+R/L\,t} = \frac{E}{L}\int e^{(R/L - a)t}\,dt + c$

$i\,e^{R/L\,t} = \frac{E}{L}\frac{e^{+(R/L-a)t}}{R/L - a} + c$

$i\,e^{R/L\,t} = \frac{E}{\cancel{L}} \cdot \frac{\cancel{L}}{R - La} e^{+(R/L-a)t} + c$

$i = \frac{E}{(R - La)} e^{(R/L - a)t} \cdot e^{-R/L\,t} + c\,e^{-\frac{Rt}{L}}$ (3m)

$i = 0$ when $t = 0 \Rightarrow 0 = \frac{E}{R - aL} + c$ (1m)

$\Rightarrow c = -E/R-aL$

$\Rightarrow i = \frac{E}{(R - La)} e^{(R/L - a)t} e^{-R/L\,t} + \frac{E}{R-aL} e^{-R/L\,t}$

$i = \frac{E}{R - La}\left[e^{-at} - e^{-R/L\,t}\right]$ (1m)

6c) $x(y')^2 - (2x+3y)y' + 6y = 0$

Put $y' = p \Rightarrow xp^2 - (2x+3y)p + 6y = 0$

$xp^2 - 2xp - 3yp + 6y = 0 \Rightarrow xp(p-2) - 3y(p-2) = 0$

$(xp - 3y)(p-2) = 0$ (1m)

$xp - 3y = 0 \Rightarrow x\frac{dy}{dx} - 3y = 0 \Rightarrow x\frac{dy}{dx} = 3y$

$\Rightarrow \frac{dy}{y} = \frac{3}{x}dx \Rightarrow \int \frac{1}{y}dy = 3\int \frac{1}{x}dx + c$

$\log y = 3\log x + \log c \Rightarrow \log y = \log x^3 c$

$\Rightarrow y = cx^3 \Rightarrow (y - cx^3) = 0$ (2m)

$p - 2 = 0 \Rightarrow \frac{dy}{dx} - 2 = 0 \Rightarrow \frac{dy}{dx} = 2 \Rightarrow dy = 2dx$

$\int 1\,dy = 2\int 1\,dx + c \Rightarrow y = 2x + c \Rightarrow (y - 2x - c) = 0$ (2m)

Hence General soln is

$(y - cx^3)(y - 2x - c) = 0$ (1m)

7a) (i) $7^{289} = 7^{4\times 72+1} = 7^{4k+1} \equiv 7 \bmod 10$

$\therefore$ 7 is the last digit (3m)

(ii) $135 \times 74 \times 48 \equiv 2 \times 4 \times 6 \bmod 7$

$$7)\,135\,(19 \qquad 7)\,74\,(10 \qquad 7)\,48\,(6$$

$$\begin{array}{r} 7 \\ \hline 65 \\ 63 \\ \hline 2 \end{array} \qquad \begin{array}{r} 70 \\ \hline 4 \end{array} \qquad \begin{array}{r} 42 \\ \hline 6 \end{array}$$

$135 \equiv 2 \bmod 7$, $74 \equiv 4 \bmod 7$, $48 \equiv 6 \bmod 7$

$135 \times 74 \times 48 \equiv 48 \bmod 7$

$135 \times 74 \times 48 \equiv 6 \bmod 7$ (4m)

Remainder is 6

7b) $6x \equiv 15 \bmod 21$

compairing with $ax \equiv b \bmod m$

$a = 6 \quad b = 15 \quad m = 21$

$\gcd(a, m) = (6, 21) = 3 = d$

Check d / b ie $3 | 15$

$\Rightarrow$ The given congruence has unique sol^n. (2m)

consider $6x \equiv 15 \bmod 21$

$6x - 15 = 21k \quad \Rightarrow 6x = 21k + 15.$

$\Rightarrow x = \dfrac{21 \cdot k + 15}{6}$ Put $k = 0 \Rightarrow x = 5/2 \notin \mathbb{Z}$

Put $k = 1 \Rightarrow x = \dfrac{36}{6} = 6$ (3m)

$\therefore x \equiv 6 \bmod 21$ is the required sol^n. (1m)

7c) $4(29)! + 5!$ is divisible by 31

By wilson's thm. $(P-1)! \equiv -1 \pmod P$

or $(P-2)! \equiv 1 \bmod P \qquad P = 31$ (1m)

$\Rightarrow (29)! \equiv 1 \bmod 31$ (1m)

$4(29)! \equiv 4 \bmod 31$ (1m)

$4(29)! + 5! \equiv (4 + 5!) \bmod 31$ (1m)

$4(29)! + 5! \equiv 124 \bmod 31$ (1m)

$4(29)! + 5! \equiv 0 \bmod 31$ (1m)

Remainder is zero

$\Rightarrow 4(29)! + 5!$ is divisible by 31 (1m)

8a) $x \equiv 5 \bmod 3, \quad x \equiv 2 \bmod 5, \quad x \equiv 1 \bmod 11$

$b_1 = 5 \quad b_2 = 2, \quad b_3 = 1$

$m_1 = 3 \quad m_2 = 5 \quad m_3 = 11$ (1m)

$(3, 5) = 1 \quad (5, 11) = 1 \quad (3, 11) = 1$

$M = m_1 \cdot m_2 \cdot m_3 = 3 \cdot 5 \cdot 11 = 165$

$M_k = \dfrac{M}{m_k} \qquad k = 1, 2, 3$

$M_k = \frac{165}{m_k} \Rightarrow M_1 = \frac{165}{m_1} = \frac{165}{3} = 55.$

$M_2 = \frac{165}{m_2} \Rightarrow M_2 = \frac{165}{5} = 33$ (1 m)

$M_3 = \frac{165}{m_3} \Rightarrow M_3 = \frac{165}{11} = 15$

consider $M_k x \equiv 1 \mod m_k$ $\quad K = 1,2,3$

$55 x_1 \equiv 1 \mod 3, \quad 33 x_2 \equiv 1 \mod 5, \quad 15 x_3 \equiv 1 \mod 11$

By inspection, $x_1 = 1$ $x_2 = 2$ $x_3 = 3$ are (3 m)
satisfied. Thus unique soln of the system is

$x \equiv b_1 M_1 x_1 + b_2 M_2 x_2 + b_3 M_3 x_3 \mod M$ (1 m)

$x \equiv 5\cdot 55 \cdot 1 + 2 \cdot 33 \cdot 2 + 1 \cdot 15 \cdot 3 \mod 165$

$x \equiv 275 + 132 + 45 \mod 165$

$x \equiv 452 \mod 165 \Rightarrow x \equiv 122 \mod 165$ (1 m)

is the unique soln.

8b) $7x + 3y \equiv 10 \mod 16$

$2x + 5y \equiv 9 \mod 16$

$a = 7 \quad b = 3 \quad r = 10 \quad n = 16$

$c = 2 \quad d = 5 \quad r = 9$

$(a, b, n) = (7, 3, 16) = 1$

$(a\, b\, n) \,|\, c \Rightarrow 1/2$. The system has soln.

$(ad - bc, n) = (29, 6) = 1$

The system has unique soln.

$7x + 3y \equiv 10 \mod 16$ — $\times 5$

$2x + 5y \equiv 9 \mod 16$ — $\times 3$

$35x + \cancel{15y} \equiv 50 \mod 16$

$6x + \cancel{15y} \equiv 27 \mod 16$

$29x \equiv 23 \mod 16$

$29x - 23 = 16k \Rightarrow x = \frac{16k+23}{29}$

Put $k = 0, 1, 2, 3 \ldots$

By inspection at $k = 4$, $x = \frac{87}{29} = 3 \in Z$

$\Rightarrow x \equiv 3 \bmod 16$ (3m)

Consider $7x + 3y \equiv 10 \bmod 16$

$21 + 3y \equiv 10 \bmod 16 \Rightarrow 3y \equiv -11 \bmod 16$

$\Rightarrow 3y + 11 = 16k \Rightarrow 3y = 16k - 11 \Rightarrow y = \frac{16k-11}{3}$

Put $k = 0, 1, 2, 3 \ldots$

$k = 2$, $y = \frac{32-11}{3} = \frac{21}{3} = 7 \in Z$

Thus $y \equiv 7 \bmod 16$ (3m)

we have $x \equiv 3 \bmod 16$, $y \equiv 7 \bmod 16$

8c) $2^{340} - 1$ is divisible by 31

we have Fermat's little thm:

$a^{p-1} \equiv 1 \bmod p$ $a = 2$ $p = 31$ (1m)

$2^{30} \equiv 1 \bmod 31$

$2^{340} = 2^{330} \cdot 2^{10}$ (1m)

$(2^{30})^{11} \equiv 1^{11} \bmod 31$ (1m)

$2^{330} \equiv 1 \bmod 31$

$2^{330} \cdot 2^{10} \equiv 1 \cdot 2^{10} \bmod 31$ (1m)

$2^{340} \equiv 1024 \bmod 31$ (1m)

$2^{340} \equiv 1 \bmod 31$ (1m)

$\Rightarrow 2^{340} - 1 \equiv 0 \bmod 31$ (1m)

Remainder zero

$\Rightarrow 2^{340} - 1$ is divisible by 31

9 a)

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 8 \\ 4 & 8 & 12 & 16 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to \frac{1}{2} R_2. \\ R_3 \to \frac{1}{4} R_3 \end{matrix}$$ (1m)

$$A \cong \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$ (2m)

$$A \cong \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$ (2m)

$$A \cong \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ Number of non-zero rows $= 2$ (1m)

$\therefore \rho[A] = 2.$

9 b)

$$A = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ -1 & -1 & 2 & -4 \\ 3 & 5 & -7 & -14 \end{array}\right] \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$ (1m)

$$A \cong \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & 0 & 3 & 4 \\ 0 & 2 & -10 & -38 \end{array}\right] \quad R_3 \to \frac{1}{2} R_3$$ (1m)

$$A \cong \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & 0 & 3 & 4 \\ 0 & 1 & -5 & -19 \end{array}\right] \quad R_2 \leftrightarrow R_3$$ (1m)

$$A \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & 1 & -5 & -19 \\ 0 & 0 & 3 & 4 \end{array}\right] \quad R_1 \to R_1 - R_2$$ (1m)

$$A \sim \left[\begin{array}{ccc:c} 1 & 0 & 6 & 27 \\ 0 & 1 & -5 & -19 \\ 0 & 0 & 3 & 4 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 2R_3 \\ R_2 \to 3R_2 + 5R_3 \end{array}$$ (1m)

$$A \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 27 \\ 0 & 3 & 0 & -37 \\ 0 & 0 & 3 & 4 \end{array}\right]$$ (1m)

$\Rightarrow \quad x = 27, \quad 3y = -37 \Rightarrow y = \frac{-37}{3}$

$3z = 4 \Rightarrow z = 4/3$ (1m)

Thus $x = 27$, $y = -37/3$ $z = 4/3$ is required soln.

9c) $$AX_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda_1 X_1$$ (1m)

$$AX_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \lambda_2 X_2$$ (1m)

$$AX_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = 4 \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \lambda_3 X_3$$ (1m)

$$AX_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = 3.5 \begin{bmatrix} 0.71 \\ -1 \\ 0.71 \end{bmatrix} = \lambda_4 X_4$$ (1m)

$$AX_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.71 \\ -1 \\ 0.71 \end{bmatrix} = \begin{bmatrix} 2.42 \\ -3.42 \\ 2.42 \end{bmatrix} = 3.42 \begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix} = \lambda_5 X_5$$ (1m)

$$AX_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix} = \begin{bmatrix} 2.416 \\ -3.416 \\ 2.416 \end{bmatrix} = 3.416 \begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix} = \lambda_6 X_6$$ (1m)

$$AX_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix} = \begin{bmatrix} 2.4146 \\ -3.4146 \\ 2.4146 \end{bmatrix} = 3.4146 \begin{bmatrix} 0.7071 \\ -1 \\ 0.7071 \end{bmatrix}$$ (1m)

Hence largest Eigen value is 3.4146 & corresponding Eigen vector is $[0.7071, -1, 0.7071]'$

10a) $$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right] \quad R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 3R_1$$ (2m)

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right] \quad R_3 \to R_3 - 7R_2$$ (1m)

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$$ (1m)

$\Rightarrow x + y + z = 3$ — (1), $-y + z = 4$ — (2), $-8z = -24$ — (3)

From (3) $z = -24/-8 = 3$ $\boxed{z = 3}$

From (2) $-y = 4 - z = 4 - 3 = 1 \Rightarrow \boxed{y = -1}$ (3m)

From (1) $x = 3 - 2y - z = 3 + 2 - 3 = 2 \Rightarrow \boxed{x = 2}$

Thus $x = 2$ $y = -1$ $z = 3$ is required solⁿ 1m

10b) $20x + y - 2z = 17 \Rightarrow x = \frac{1}{20}[17 - y - 2z]$ — (1)

$3x + 20y - z = -18 \Rightarrow y = \frac{1}{20}[-18 - 3x + z]$ — (2)

$2x - 3y + 20z = 25 \Rightarrow z = \frac{1}{20}[25 - 2x + 3y]$ — (3)

Put $y=0$ $z=0$ in Eqn ①

$x^{(1)} = \frac{17}{20} = 0.85$

Put $x = 0.85$ $z=0$ in Eqn ②

$y^{(1)} = \frac{1}{20}[-18-3(0.85)+0] = -1.0275$

Put $x=0.85$ $y=-1.0275$ in Eqn ③

$z^{(1)} = \frac{1}{20}[25-2(0.85)+3(-1.0275)] = 1.0109$ (2m)

This completes first iteration

Put $y = -1.0275$ $z = 1.0109$ in Eqn ①

$x^{(2)} = \frac{1}{20}[17-(-1.0275)+2(1.0109)] = 1.0025$

Put $x = 1.0025$ $z = 1.0109$ in Eqn ②

$y^{(2)} = \frac{1}{20}[-18-3(1.0025)+1.0109] = \cancel{-}0.9998$

Put $x = 1.0025$ $y = -0.9998$ in Eqn ③

$z^{(2)} = \frac{1}{20}[25-2(1.0025)+3(-0.9998)] = 0.9998$

This completes second iteration (2m)

Put $y = -0.9998$ $z = 0.9998$ in Eqn ①

$x^{(3)} = \frac{1}{20}[17-(-0.9998)+2(0.9998)] = 0.99997 \simeq 1$

Put $x = 0.99997$ $z = 0.9998$ in Eqn (2)

$y^{(3)} = \frac{1}{20}[-18-3(0.99997)+0.9998] = -1.0000055 \simeq -1$

Put $x = 0.99997$ $y = -1.0000055$ in Eqn (3)

$z^{(3)} = \frac{1}{20}[25-2(0.99997)+3(-1.0000055)] = 1.0000022$
$\simeq 1$

(2m)

Thus $x \simeq 1$ $y \simeq -1$ $z \simeq 1$ is required soln.

10c) $x + 2y - z = 1$, $2x + y + 4z = 2$
$3x + 3y + 4z = 1$

```
A = np.matrix([[1 2 -1], [2 1 4], [3 3 4]])
B = np.matrix([[1], [2], [1]])                                   (1m)
~~AB = np.linalg.matrix_rank(A)~~
AB = np.concatenate((A,B), axis=1)
rA = np.linalg.matrix_rank(A)
rAB = np.linalg.matrix_rank(AB)                                  (1m)
n = A.shape[1]
if (rA == rAB):
if (rA == n):                                                    (1m)
print ("The system has unique soln")
print (np.linalg.solve(A,B))
else:
 print ("The system has infintely many soln")                    (1m)
else:
 print ("The system of equations is
         inconsistent")
```

Out : The system has unique soln

```
[ [7.]
  [-4.]
  [-2.] ]                                                        (1m)
```

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